The Property of Phase Velocity in the Cloak Shell

Jing-jing FU¹, Guan-xia Yu²*, Min LUO², Wen-wen DU¹ and Jun-yuan DONG¹

¹College of Information Science and Technology, Nanjing Forestry University, Nanjing 210037, P. R. China
²College of Science, Nanjing Forestry University, Nanjing 210037, P. R. China
*Corresponding author

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Abstract. According to classical EM field theory, the dispersion relation for anisotropic media in the 2-D cylinder cloak have been calculated in the KDB system (consisting of k vector and the DB plane). Due to the special and variable Electromagnetic (EM) parameters, theoretical results indicate that the phase velocities of EM waves are non-uniform in the cloak shell. The numerical simulations show that the phase velocity is reduced, and less than that of light in vacuum in the front and rear of cloak shell. While in the upper and lower region of cloak shell, the phase velocities are larger than that of other regions, and exceed that of light in vacuum.

Introduction

Recently, artificially structured metamaterials have attracted worldwide attention due to their unprecedented flexibility in manipulating electromagnetic (EM) waves and producing new functionalities[1-5]. One of the novel features is that metamaterials can produce negative permeability and permittivity simultaneously. Due to the negative refraction, there are lots of unusual properties and potential applications, such as the superlens, the energy localization, the super waveguide, and so on[1,6-7]. Besides negative refraction and cloaks, more wave manipulation strategies using metamaterials have been drawn out, such as EM field concentration, rotators, direction changing, illusion optic, and so forth[8-14].

We all know that the novel properties of metamaterials in the electromagnetic fields are due to the flexible, variable and anisotropic electromagnetic parameters, and the special materials can be designed and fabricated by man-made periodic and not be found in nature materials, which make up for lack of nature materials. There are many unknown fundamental processes in physics when EM waves transmit through the metamaterials. In order to comprehend and design the novel devices, it is inevitable to know the propagation properties of EM waves in the metamaterials. The investigation of transmitting properties of EM waves in metamaterials not only have extensive applications of the metamaterials in the military, communication and antenna, but also enrich the classical theory of electromagnetic field.

In this paper, we have focused on the propagation characteristics of the electromagnetic waves in the general anisotropic medium, and investigated the distribution of phase velocities in the cloak shell. Firstly, the dispersion relation for general anisotropic medium has been deduced in the KDB system[15] based on classical EM fields theory. Secondly, the phase velocity of in the general anisotropic mediums has been derived and analyzed theoretically. Thirdly, the distribution of the velocity in cloak shell has been investigated in numerical methods.

Theoretical Analysis

For simplicity, we restrict the problem to 2-D cylindrical cloak in which EM fields are excluded from an infinite circular cylinder with radius R₁, as shown in Figure 1. The cloaking region is consisted of a concentric cylindrical shell with inside radius R₁ and outside radius R₂, which is filled with the following radius-dependent, anisotropic relative permittivity and permeability (in rectangular coordinates):
\[ \varepsilon_r = \begin{bmatrix} \varepsilon_{xx} & \varepsilon_{xy} & 0 \\ \varepsilon_{yx} & \varepsilon_{yy} & 0 \\ 0 & 0 & \varepsilon_{zz} \end{bmatrix} \]  
\[ \mu_r = \begin{bmatrix} \mu_{xx} & \mu_{xy} & 0 \\ \mu_{yx} & \mu_{yy} & 0 \\ 0 & 0 & \mu_{zz} \end{bmatrix} \]

where \( \varepsilon_{xx} = \frac{1}{\varepsilon_0} \left( \frac{\nu^2 - \kappa^2}{\nu^2} \right) = \frac{\nu^2 - \kappa^2}{\nu^2} \), \( \varepsilon_{yy} = \frac{1}{\varepsilon_0} \left( \frac{\kappa^2 - \nu^2}{\kappa^2} \right) = \frac{\kappa^2 - \nu^2}{\kappa^2} \), and \( \varepsilon_{zz} = \frac{1}{\varepsilon_0} \left( \frac{\kappa^2 - \nu^2}{\kappa^2} \right) = \frac{\kappa^2 - \nu^2}{\kappa^2} \). The cloak shell separate space into three region: free space (region 1), cloak shell (region 2), sheltered region (region 3).

\[
T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos \theta & -\sin \theta \\ 0 & \sin \theta & \cos \theta \end{bmatrix}
\]

The relation between a vector \( \mathbf{A} \) in the rectangular coordinate system and the same vector \( \mathbf{A}_k \) in the KDB system is governed by \( \mathbf{A}_k = T \mathbf{A} \), where subscript \( k \) represents a vector in the KDB system. Therefore, the relation of field vectors in two different systems can be expressed as:

\[ \bar{\mathbf{E}}_i = \tilde{T}_i \bar{\mathbf{D}}_i \quad \bar{\mathbf{H}}_i = \tilde{T}_i \bar{\mathbf{B}}_i \]

where \( \varepsilon_0 \) and \( \mu_0 \) are permittivity and permeability in a vacuum, respectively. \( \kappa \) and \( v_k \) are quantities related to relative permittivity and permeability of anisotropic medium in the KDB system, which are defined as: \( \kappa = T \kappa T^{-1}, v_k = T v T^{-1} \), where \( T^{-1} \) is the inverse of \( T \). In our system, \( \kappa \) and \( v_k \) can be written as:

\[ \kappa' = \begin{bmatrix} \kappa_{xx} & \kappa_{xy} & 0 \\ \kappa_{yx} & \kappa_{yy} & 0 \\ 0 & 0 & \kappa_{zz} \end{bmatrix}, v_k = \begin{bmatrix} v_{xx} & v_{xy} & 0 \\ v_{yx} & v_{yy} & 0 \\ 0 & 0 & v_{zz} \end{bmatrix} \]

in which \( e_{kxx} = e_{xx}/e_{kxx} = e_{kyy} = e_{yy}/e_{kxx} = 1/e_{kxx} \), and \( m = e_{kxx} e_{kyy} - e_{kxy}^2 \).

Within the frame of KDB system, the Maxwell equations for the TE plane waves take the same form as those in the rectangular coordinate system. The EM fields components of \( \mathbf{e}_1 \) and \( \mathbf{e}_2 \) direction can be rearranged and written in the matrix form:

\[ \begin{bmatrix} \kappa_{xx} & \kappa_{xy} & 0 \\ \kappa_{yx} & \kappa_{yy} & 0 \\ 0 & 0 & \kappa_{zz} \end{bmatrix} \begin{bmatrix} D_{x1} \\ D_{y1} \\ D_{z1} \end{bmatrix} = \begin{bmatrix} \kappa_{xx} & \kappa_{xy} & 0 \\ \kappa_{yx} & \kappa_{yy} & 0 \\ 0 & 0 & \kappa_{zz} \end{bmatrix} \begin{bmatrix} V_{x1} \\ V_{y1} \\ V_{z1} \end{bmatrix} \]

where \( \omega \) is angular frequency. Eliminating \( B_k \) or \( D_k \) in the equations(5), and defining \( c_0 = 1/(\mu_0 \varepsilon_0)^{0.5} \).
(the speed of light in a vacuum), the dispersion relation for anisotropic media can be obtained:

\[
\omega^4 - c_0^2 (a_k k_z^2 + b_k k_z^2) \omega^2 + c_0^4 (c_k k_z^4 + d_k k_z^4 + e_k k_z^4) = 0
\]  \hspace{1cm} (6)

where \(k_z = k \cos \theta\) and \(k_s = k \sin \theta\) are wave numbers in z-direction and transverse direction, respectively, and \(a_k = 2(e_{zz} - e_{xx}) - 2e_{xy}, b_k = e_{xx}, c_k = 2(e_{zz} - e_{xx}), d_k = (e_{xx} e_{zz})^{1/2}, \) and \(e_k = 2(e_{zz} - e_{xx})(e_{xx} e_{zz})^{1/2}\). For given angular frequency \(\omega\), there are four solutions for \(k\) in the equation (9), which are corresponding to the type 1 wave and type 2 wave of forward propagation and backward propagation in the general anisotropic media. If the \(e_{xy} = e_{yx} = 0\) in the formula 1, the general anisotropic media is reduces to biaxial anisotropy medium. The equation (9) can be simplified as follows:

\[
[\omega^2 - c_0^2 (\frac{1}{e_{xx}} + \frac{1}{e_{yy}})] [\omega^2 - c_0^2 (\frac{1}{e_{xx}} + \frac{1}{e_{yy}})] = 0
\]  \hspace{1cm} (7)

This equation can be decomposed into the two familiar dispersion relations of TE mode and TM mode waves. Applying phase velocity \(u = \omega/k\) and formula (7), the phase speed in the anisotropic media can be obtained:

\[
v^4 - \left(\frac{c_0}{k}\right)^2 (a_k k_z^2 + b_k k_z^2) v^2 + \left(\frac{c_0}{k}\right)^4 (c_k k_z^4 + d_k k_z^4 + e_k k_z^4) = 0
\]  \hspace{1cm} (8)

**Simulation Results**

In order to calculate the phase velocity in cloak and around medium, full-wave simulations have been carried out. We choose a square calculating region with \(-2m \leq x \leq 2m\) and \(-2m \leq y \leq 2m\), and a 2-D cylindrical cloak with an inner radius \(R_1 = 0.8m\) and outer \(R_2 = 1.1m\) is located at the center of square region. A plane wave is incident from left boundary of calculating region with frequency \(f = 1GHz\).

The equation (11) shows the different anisotropic parameters decide different phase velocities in the medium. According to formula (1), the EM parameters in the cloak are the functions of the positions, then the phase velocities in the cloak shell vary with positions.

**Figure 2.** (a)(b) Distribution of electric field \(E_z\) and phase velocity in the x-y plane, respectively, where inner radius \(R_1 = 0.8m\), outer radius \(R_2 = 1.1m\), and \(f = 1GHz\).

**Figure 3.** (a) The distribution of velocity along y-direction at \(x = 1m\). (b) The distribution of velocity along x-direction at \(y = 1m\). The other conditions are the same as Figure 3.
If a plane wave is incident from free space into a cloak, as we know, the impendence of metamaterials in the cloak shell, which parameters are acquired by method of transformation optics, match with surrounding materials (free space) in essence. Thus, for the TE-polarized incident wave \( E_z = E_0 e^{ik_0 z} \), the z-components of the electromagnetic field in the region 1 (free space: isotropic media) and region 2 (cloak shell: anisotropic media) in the cylindrical coordinates can be written in the following forms:

\[
E_{z1} = \sum_{n=-\infty}^{\infty} i^n E_n J_n(k_n r)e^{i\theta}
\]

\[
E_{z2} = \sum_{n=-\infty}^{\infty} i^n E_n J_n(mk_n r)e^{i\theta}
\]

(9)

where \( k_0 \) is wave number in the free space, \( J_n \) is n-order bessel function, and \( m=R_2/(R_2-R_1) \). Figure 2a shows the distribution of electric field \( E_z \) of TE mode EM field, where we choose \( E_0=1 \). When the plane wave passes through the region of cloak, the wave is compressed in the cloak shell, and diffracted smoothly the cloak region. Because the impendence of metamaterial in the cloak region is matched with around media, there is no reflected wave from cloak shell. Therefore, the shape of plane wave can be maintained when the plane wave is distorted by the shelved object in the center of circular cylinder, but the EM wave is well guided and diffracted smoothly through the cloak shell by the special metamaterial.

In free space, as is well-known, the phase velocity is uniform, and equal to the speed of light in vacuum \( 3 \times 10^8 \) m/s. While in the cloak shell, the equiphase surface is not a plane, that is, the phase velocities in the same equiphase surface are not equivalence and non-uniform distribution. According to formula (8), the distribution of phase velocities in the x-y plane are shown in the Figure (2b). In the front and the rear of cloak shell, the EM waves are compressed in the cloak shell, and the equiphase surface lag behind those in the free space. At the same time, in the two regions, the phase velocity is reduced, and less than that of the the velocity of light in vacuum. Figure 3a shows that the distribution of phase velocity along y-direction at \( x=1 \) m. Between \( y=0.5 \) m and \( y=0.5 \) m, the phase velocity is only 0.6-0.7 times speed of light in vacuum. We also can see lower the phase velocity is, the closer to outer boundary the positions are in the front and the rear of cloak shell (Figure (2b)). In the middle region (the upper and the lower of circle cylinder along direction of propagation) of cloak shell, the phase velocities are larger than that of other region, and exceed that of light in vacuum (Figure 2b). In the region, the phase velocity is sped up to make up the velocity loss in the front and rear of cloak shell. Figure 3b shows that the distribution of phase velocity along x-direction at \( y=1 \) m. Between \( x=-0.5 \) m and \( x=0.5 \) m, the phase velocity reaches 1.5-3.5 times velocity of light in vacuum. Also, the phase velocity is not non-uniform distributions, and the phase velocity increase sharply near the inner boundary of cloak shell (Figure (2b)).

Although the superluminal phenomena is rare in the normal nature medium, the special phenomena can be appeared in the anomalous dispersion medium. Based on Einstein’s special theory of relativity, the speed of a moving object cannot exceed that of light in vacuum. Because the speed of moving object in the Einstein’s special theory of relativity is related to the speed of energy flow, the superluminal phenomena of phase velocity in the anomalous dispersion medium is not inconsistent with it. In the isotropic nondispersive media, the phase velocity of EM waves can be simplified as: \( v = 1/(\mu \varepsilon)^{0.5} \), in which \( \varepsilon \) and \( \mu \) are the EM parameters of medium. Because EM parameters in the free space is less than that of other media, the velocity of the phase in the general media is less than the velocity of light in vacuum. Metamaterial as a special man-made materials, has variable EM parameters (Formula (1)), which can be designed periodically or aperiodically. The special variable EM parameters can be larger or less than the values of nature materials, even equal to zero or less than zero. According to classical theories on electromagnetic field, the less EM parameters are, the larger of the phase velocity and energy flow of EM waves are. Thus, when the EM parameters in the metamaterials are less than that of nature materials, the faster-than-light EM waves in the metamaterials are common phenomena.
Conclusion
In this paper, the dispersion relation and the phase velocity for anisotropic media in the 2-D cylinder cloak have been calculated in the KDB system according to classical EM field theory. The theoretical analysis and simulations show that there is superluminal phenomena in the cloak shell consisting of metamaterials. In the upper and the lower region of cloak shell, the phase velocities are larger than that of other region, and exceed that of light in vacuum. While in the front and the rear of cloak shell, the phase velocity is reduced and less than that of light in vacuum. The investigation of transmitting properties of EM waves in metamaterials not only have extensive applications in the militray, communication and antenna, but also enrich the classical theory of electromagnetic field.

References