From Initial to Final State: Single Step Prediction of Structural Dynamic Response

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ABSTRACT

Accurately predicting the structural response under dynamic loads is of great importance to evaluate the structure's performance, monitor the structure's health and control the vibration. The development of a real-time prediction method is challenging because the model must have a low computational burden and high running speed. Due to the computing convergence requirements and complexity constraints, traditional numerical methods such as the finite element method and the finite difference method must take thousands to millions of short-time steps to calculate the final structural dynamic response we really want. After costing a lot of computing resources and time, these internal steps are worthless but significantly reduced the efficiency of numerical methods. To tackle this problem, in this paper, we propose a single-step method named physics-informed implicit Runge-Kutta (PI-IRK) to predict the structure dynamic response straightly from the initial to the final state. Specifically, we fuse discrete-time physics-informed neural networks (PINNs) and implicit Runge–Kutta method with low-expense hide stages. In the proposed method, deep neural network models are employed as the core to predict the Runge-Kutta stages and the final state. We integrate physics information such as implicit Runge–Kutta form of structure vibration governing equation and boundary constraints as the prior information into the neural networks model. With the assistance of the prior information, the proposed PI-IRK model is an unsupervised learning model that can be trained without any measurement data. Without any internal steps, the PI-IRK model can straightly predict the final structural dynamic response after training. The accuracy of the proposed method is demonstrated by predicting the structural response of a cantilever beam under a distributed dynamic load even with a large time step.

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1 INTRODUCTION

In structural dynamics, prediction of structural dynamic response is a key step to analyze the stability, vibration features and monitor the structural health. Numerical simulation methods have been widely used in the prediction of structural response of engineering systems. A large number of engineering cases have been successfully implemented in recent decades to analyze the stability of structures with numerical simulation to ensure their design safety. For example, the finite element method, after decades of development, has become a mature commercial numerical simulation method and is popular in engineering applications, such as: civil engineering [1], fluid mechanics [2], thermodynamics [3] and aerodynamics [4]. In modern large-scale and complex structural engineering, the nonlinear behavior in materials and structures dramatically increases the complexity of numerical simulation methods, which is still a heavy task even in large computing resource clusters. In addition, due to the limitation of the stability, the forward calculation process of the numerical simulation method relies on thousands or even millions of time steps to ensure the stability and convergence of the solution.

In order to overcome the above difficulties in numerical simulation methods, researchers turn their attention to the rapidly developing artificial intelligence. By exploring the establishment of meta-models to describe the input-output relationships of simulation systems, researchers strive to implement surrogate models to replace the time-consuming numerical simulation methods. In the meta-model, some machine learning methods such as Bayesian regression [5], Gaussian process [6, 7] and support vector machine [8] have been successfully applied in many engineering problems. However, most of these machine learning algorithms are supervised learning, which relies on a large amount of training data to train the model. In the prediction of structural dynamic response, since the future state is unknowable and unmeasurable, we cannot obtain the accurate future structural response to train our supervised learning algorithm.

To break the limitation of training data, a novel machine learning algorithm called physics-informed machine learning [9] (PIML) has been proposed to calculate solutions of various physics problems. By using physical information, such as governing equations and boundary conditions, as prior information to train machine learning algorithms, the training data requirements of PIML are significantly reduced. The PIML method has been successfully applied in engineering fields, such as physics-informed neural networks in fluid mechanics [10] and thermodynamics [11], physics-informed Gaussian process in track dynamics [12] and physics-informed long short-term memory model in structural dynamics [13].

In this paper, we fuse the idea of PIML with implicit Runge-Kutta (IRK) to propose the method of physics-informed implicit Runge-Kutta (PI-IRK) to predict the structural dynamic response in single step. IRK is an excellent method to solve ordinary/partial differential equations due to its excellent stability, error control and adaptability to large time step. But how to quickly solve a series of nonlinear equations of implicit stages in IRK method is still a challenge. In this paper, we try to utilize a deep neural network to approximate the implicit stages and final solution of IRK. We use the nonlinear equations between the implicit stages in IRK as prior physics information to train the neural network model, so we can transform the problem of solving the nonlinear equations into an optimization problem of neural network loss function. Due to the stability of the IRK method for long time steps, we can obtain the final state directly from the initial state without any internal time steps. In PI-IRK, we exploit the automatic differentiation of neural networks to compute the spatial differentiation in the governing equations without any discretization errors. Through a case of cantilever beam vibration, we prove that our proposed method can obtain satisfactory accuracy by utilizing a gradient-based optimizer to optimize the loss function of the neural network.

2 METHOD

2.1 Runge-Kutta Method

The Runge-Kutta method is a classical method to solve ordinary/partial differential equations. Let's consider an ordinary differential equation as:

$$y' = f(t), t > t_0 \tag{1}$$

with initial condition as:

$$y(t_0) = y_0 \tag{2}$$

If we know the solution y_n at time step n, we can calculate the next time step solution y_{n+1} as:

$$\xi_j = y_n + h \sum_{i=1}^{v} a_{j,i} f(t_n + c_i h, \xi_i), \quad j = 1, 2, \cdots, v \tag{3}$$

$$y_{n+1} = y_n + h \sum_{j=1}^{v} b_j f(t_n + c_j h, \xi_j) \tag{4}$$

Here $\boldsymbol{A} = (a_{j,i})_{, j,i=1,2,\cdots,v}$ is the RK matrix, $\boldsymbol{b} = [b_j]_{, j=1,2,\cdots,v}$ is the RK weights vector and $\boldsymbol{c} = [c_j]_{, j=1,2,\cdots,v}$ is the RK nodes vector. v, h is the order and step time of RK method respectively. $\xi_{j, j=1,2,\cdots,v}$ are the RK stages.

If the $\boldsymbol{A}$ matrix is strictly lower triangular, where each ξ_j only depends on the previous RK stages, the RK method is called the explicit Runge-Kutta method. If the $\boldsymbol{A}$ matrix is not strictly lower triangular, where ξ_j depends on each other, it is necessary to solve a series of nonlinear equations to obtain the values of ξ_j, then the method is called the implicit Runge-Kutta method.

2.2 PI-IRK Method

Due to the excellent stability of the IRK method and its adaptability to large time step, we choose the IRK method as our basic method to predict the dynamic response of structures. But for partial differential equations in structural dynamics, which usually involve higher-order partial derivatives, it is too complicated for IRK method to solve a series of nonlinear equations involving higher-order partial derivatives to obtain accurate RK stages. On the other hand, PINN [14] has been proven to be an effective framework to solve nonlinear partial differential equations, so we introduce the idea of PINN to solve the nonlinear equations in the IRK method to obtain the RK stages.

Firstly, let us consider a structural dynamics equation as:

$$m(x)\,\ddot{u}(x,t) \;+\; c(x)\,\dot{u}(x,t) \;+\; k(x)u(x,t) = f(x,t) \tag{5}$$

where x is the number of degrees of freedom for multi-degree-of-freedom systems or spatial coordinates for continuum structures. m, c, k are mass, damping and stiffness coefficients at x respectively. $f(x,t)$ is the external force on x at time t. $u, \dot{u}, \ddot{u}$ are the displacement, velocity and acceleration responses of the structure respectively. We know that the current state (u_0, v_0) of the structure system and our target is to predict the structural response (u_t, v_t) of the structure with time step h.

We build the s-order IRK form of the structural dynamic equation as:

$$v_j = v_0 + \frac{h}{m(x)} \sum_{i=1}^{s} a_{j,i} \cdot \left(f(x, t_0 + c_i h) \;-\; k(x) \cdot u_j - c(x) \cdot v_j\right), \;\; j = 1, 2, \cdots, s \tag{6}$$

$$u_j = u_0 + h \sum_{i=1}^{s} a_{j,i} \cdot v_j\,, \;\; j = 1, 2, \cdots, s \tag{7}$$

$$v_t = v_0 + \frac{h}{m(x)} \sum_{i=1}^{s} b_j \cdot \left(f(x, t_0 + c_i h) \;-\; k(x) \cdot u_j - c(x) \cdot v_j\right) \tag{8}$$

$$u_t = u_0 + h \sum_{i=1}^{s} b_j \cdot v_j \tag{9}$$

Here, (u_j, v_j) is the j-th order RK step of (u, v) which is obtained by solving the nonlinear equations of Eq. (6-7). Then we can predict the structural response (u_t, v_t) with Eq. (8-9). However, it is too complicated to solve these nonlinear equations, especially when $k(x)$ or $f(x,t)$ in the equation involves nonlinearity. For describing such complex nonlinear relationships, there are great advantages in deep neural network models. By defining two multi-output fully connected neural networks model with input x and output $[o_1^v, o_2^v, \cdots, o_s^v, o_t^v]$, $[o_1^u, o_2^u, \cdots, o_s^u, o_t^u]$ respectively, we can use the two model to approximate the RK steps and (u_t, v_t) in Eq. (6-9).

According to the physical information of the s-order IRK method described by Eq. (6-9), the loss function of the governing equation of neural network is calculated as:

$$loss_f = \sum_{j=1}^{s} \sum_{i=1}^{N_f} \left|o_j^v(x_i) - v_j(x_i)\right|^2 + \sum_{j=1}^{s} \sum_{i=1}^{N_f} \left|o_j^u(x_i) - u_j(x_i)\right|^2 + \sum_{i=1}^{N_f} |o_t^v(x_i) - v_t(x_i)|^2 + \sum_{i=1}^{N_f} |o_t^u(x_i) - u_t(x_i)|^2 \tag{10}$$

where N_f is the number of degrees of freedom or the number of collocation points. For systems with discrete degrees of freedom, the neural network input x is the number of degrees of freedom, while for continuous deformation systems such as beam and plate, the neural network input x is the coordinates of collocation points, which are randomly or equidistantly sampled in the defined spatial domain. These collocation points are different from the nodes in the numerical method. We do not discretize the continuous deformation system, and we can predict the structural response not only at collocation points but at any points in the defined spatial domain. These collocation points can also vary during the calculation process.

In addition, there are boundary conditions in the structural dynamics, such as the displacement constrained at the boundary, we can also construct the loss function of the boundary condition of the neural network as:

$$loss_b = \sum_{j=1}^{s} \left| o_j^u(x_b) - u_j(x_b) \right|^2 + \left| o_t^u(x_b) - u_t(x_b) \right|^2 \tag{11}$$

Where, x_b is the point on the boundary condition, $u(x_b)$ is the displacement constrained at the boundary. The weighted sum of these loss functions as Eq. (12) can be used as the total loss function of the neural network. By employing an optimizer to minimize the $loss_{total}$, the output of the neural network can satisfy the constraint of Eq. (6-9) gradually, so that the output (o_t^v, o_t^u) can approach the exact solution.

$$loss_{total} = w_f \cdot loss_f + w_b \cdot loss_b \tag{12}$$

For continuous system vibration problems, which mostly involve the partial derivatives of u in space, we can use the automatic differentiation [15] of neural networks to calculate these partial derivatives instead of numerical differentiation. Since the neural network can be regarded as a series of operations of matrices and nonlinear activation functions, if the nonlinear activation function is differentiable, automatic differentiation can calculate the analytical derivative of neural network output u to input x along the backpropagation chain of the neural network without discretization errors.

3 NUMERICAL EXPERIMENT

This experiment aims to highlight the ability of the proposed PI-IRK model to predict the dynamic response of a continuum system. To this end, let us consider a continuous beam with one end fixedly constrained and the other end vibrating freely. The governing equation of this cantilever beam is:

$$\rho A \frac{\partial^2 u}{\partial t^2} + c \frac{\partial u}{\partial t} + EI \frac{\partial^4 u}{\partial x^4} = f(x,t) \tag{13}$$

with boundary conditions as:

$$u(0,t) = \frac{\partial u(0,t)}{\partial x} = \frac{\partial u^2(x_0,t)}{\partial x^2} = \frac{\partial u^3(x_0,t)}{\partial x^3} = 0 \tag{14}$$

where, $\rho A, c, EI, f(x,t)$ are the beam mass, damping, stiffness coefficient and external force respectively. The length of the beam is x_0. We consider the length of the cantilever beam to be 1m. The initial state of the beam is undeformed and static, and it deforms under the external load $f(x,t) = 100 \cdot sin(t) \cdot x$. We set the $\rho A = 10$, $\mathrm{c} = 0.1$ and $EI = 100$. With the initial state of the beam, we wish to predict its structural response state at 5s. We build the IRK form of this problem as:

$$v_j = v_0 + \frac{h}{\rho A} \sum_{i=1}^{s} a_{j,i} \cdot \left(f(x, t_0 + c_i h) - EI \cdot \frac{\partial^4 u_j}{\partial x^4} - c \cdot v_j \right), \ \ j = 1, 2, \cdots, s \tag{15}$$

$$u_j = u_0 + h \sum_{i=1}^{s} a_{j,i} \cdot v_j\,, \quad j = 1, 2, \cdots, s \tag{16}$$

$$v_t = v_0 + \frac{h}{\rho A} \sum_{i=1}^{s} b_j \cdot \left(f(x, t_0 + c_i h) - EI \cdot \frac{\partial^4 u_j}{\partial x^4} - c \cdot v_j\right) \tag{17}$$

$$u_t = u_0 + h \sum_{i=1}^{s} b_j \cdot v_j \tag{18}$$

Here we take h to be 5s, which means that from the initial state we directly predict the final state in single time step without any internal time step. In traditional numerical methods, the time step is usually set to be very small to ensure the stability and accuracy of the algorithm. The PI-IRK method allows us to take a time step of 5s while still retaining stability and high predictive accuracy.

We consider a PI-IRK method of order 10. We firstly establish two neural networks models with 4 hidden layers and 20 neurons per layer. Each model has 11 outputs to approximate RK steps and the final state. The training set consists of $N_f = 101$ collocation points equidistantly sampled over the whole beam. So, the loss function $loss_f$ of the governing equation is calculated by Eq. (10). In addition, the $loss_b$ of the boundary condition is calculated as Eq. (11).

The total loss function $loss_{total}$ is calculated as Eq. (12) with $w_f = 1$ and $w_b = 100$. We employ a gradient-based optimizer Limited-memory Broyden–Fletcher–Goldfarb–Shanno (LBFGS) algorithm to minimize the $loss_{total}$. After 1000 iterations with a learning rate of 0.1, we obtain the dynamic response prediction of the cantilever beam at $t = 5s$ as shown in Figure 1. From the initial state directly to the final state, our PI-IRK method utilizes only single time step to predict the dynamic response of the

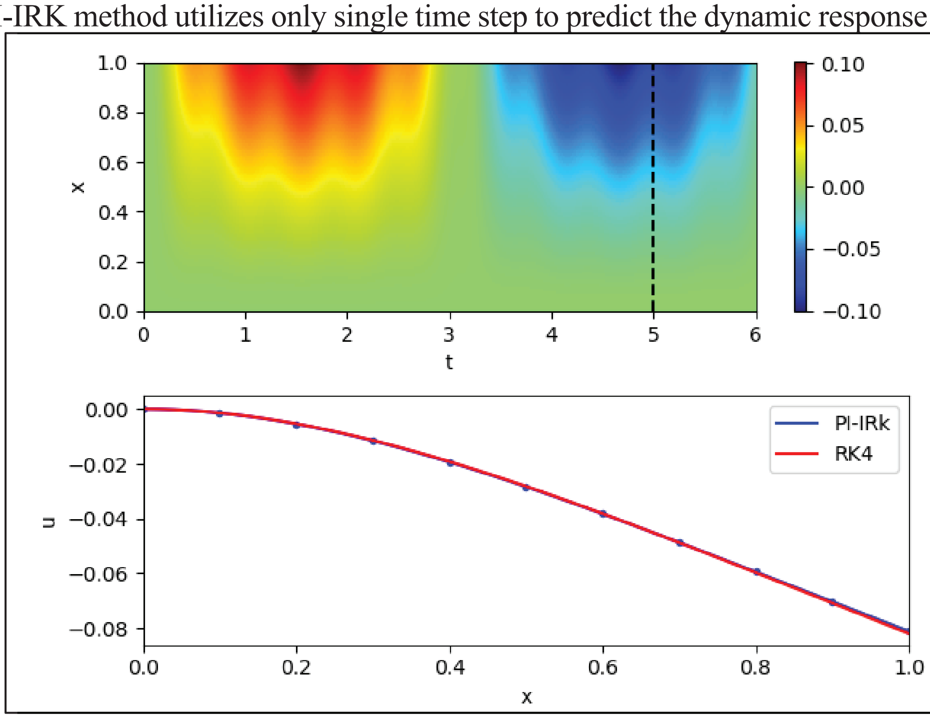

Figure 1. Vibration of the cantilever beam: Top: Benchmark solution obtained by explicit 4-order Runge-Kutta method. Bottom: PI-IRK solution and benchmark solution at time t=5s

structure with an L2 relative error of 0.00635.

4 CONCLUSIONS

We have introduced the physics-informed implicit Runge-Kutta (PI-IRK) model, a mesh-free single-step method to predict the structural dynamic response from the initial state to the final state. In the proposed method, the neural network model is used to approximate the Runge-Kutta steps of the multi-order implicit Runge-Kutta to transform the problem of solving complex nonlinear equations into the problem of minimizing the loss function of the neural network. In addition, due to the continuous prediction of the neural network, the proposed method can predict the structural vibration response at any spatial position in the system without interpolation. Through a case of continuous beam vibration, we demonstrate the accuracy of the proposed method.

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