Bayesian Damage Estimation with Regularized Data-Driven Stochastic Time Series Model

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ABSTRACT

A probabilistic vibration-based global SHM technique is proposed. In the process, experimental data from a modal test on a wing structure is used to identify a unified model with i) a Vector-dependent Functionally Pooled (VFP) component, ii) and an Auto-Regressive eXogenous (ARX) component. LASSO regularization is incorporated as a model structure selection method while introducing model sparsity. A probabilistic damage identification/quantification method within a Bayesian architecture is applied to solve the inverse problem, which provides a decision confidence interval for damage estimation.

INTRODUCTION

Engineering structures are subject to many sources of uncertainty; from varying operating/environmental conditions to complex damage evolution, time-varying dynamics, and nonlinear behavior in seemingly identical components. Vibration-based active-sensing Structural Health Monitoring (SHM) methods form an important family of SHM methods that are frequently based on statistical/probabilistic metrics and damage-sensitive features developed for accurately and robustly analyzing complex structural dynamics and allowing the extraction of decision confidence intervals. To this end, stochastic time series models and derivative methods have been extensively used within vibration-based SHM to overcome the above challenges. Advantages such as accuracy in modeling system dynamics, robustness against uncertainties, and low data footprint make these models attractive for SHM applications. On the other hand, the damage state estimation task (localization and quantification) constitutes an inverse problem whose effective treatment depends on the forward system identification process (selected model structure, model orders, parameter estimation, and potential hyperparameter tuning) and cor-

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responding statistical properties of the selected model.

Regularization has been used extensively to overcome ill-posed linear inverse problems or to force sparsity when the model suffers poor generalization [1, 2]. Within the framework of linear system identification, regularization posts physically comprehensible effects on model dynamics over time, space, and frequency domains. l^1-norm regularization (e.g. LASSO) induces sparsity by eliminating model parameters with a small impact. Thus, sparse model identification techniques have been presented to increase model generalizability and discover true dynamics [3, 4].

Regarding the inverse problem of damage identification, deterministic approaches were used in traditional approaches, where the estimation result points to a specific damage state without providing any statistical information [5–7]. Previous work done on VFP-STS framework with GA-SQP inverse estimation managed to leverage the STS model residual to provide probabilistic damage estimation with a rather deterministic inverse technique. The Bayesian approach has been used extensively with deterministic forward models such as finite element (FE) and constitutive models [8, 9]. By adopting the Bayesian approach and probabilistic sampling in inverse/optimization problems, probabilistic damage diagnosis can be achieved independently over the STS model and obtain the posterior distribution of both the damage parameter and STS model residual.

In order to avoid ill-posed and ill-conditioned inverse problems in damage state estimation, this paper introduces the LASSO regularization to VFP-STS model as a model structure selection technique to overcome overfitting and improve model generalization capability. Additionally, the Bayesian inversion (MCMC-AM) is applied to the inverse problem to account for out-of-sample uncertainties and provided a probabilistic damage estimation results.

LASSO REGULARIZED VFP-ARX

The estimation of VFP-ARX model follows the system identification process introduced by [10,11]. Via VFP method, multiple ARX models can be treated as one entity in model identification. The stochasticity in the data set is characterized as time series residual covariance and related to damage state vector $\boldsymbol{k}$ via functional dependency [12, 13]. The general form of VFP-ARX$(na)_p$ model is given by:

$$y_{\boldsymbol{k}}[t] + \sum_{i=1}^{na} a_i \cdot y_{\boldsymbol{k}}[t-i] = \sum_{i=0}^{nb} b_i \cdot x_{\boldsymbol{k}}[t] + e_{\boldsymbol{k}}[t] \qquad e_{\boldsymbol{k}}[t] \sim \text{ iid}\,\mathcal{N}\big(0, \sigma_e^2\big) \tag{1}$$

$$a_i(\boldsymbol{k}) = \sum_{j=1}^{p} a_{i,j} \cdot G_j(\boldsymbol{k}), \quad b_i(\boldsymbol{k}) = \sum_{j=1}^{p} b_{i,j} \cdot G_j(\boldsymbol{k})$$

where $na = nb$ designating the model order, p the number of function basis. with $y_{\boldsymbol{k}}[t]$ the data under various states specified by state vector $\boldsymbol{k} = [k_1,\ k_2, \ldots, k_n]$. $e_{\boldsymbol{k}}[t]$ is the residual sequence of the model, which is assumed a white (serially uncorrelated) zero mean sequence with variance $\sigma_e^2(\boldsymbol{k})$. $G_j(\boldsymbol{k})$ is the function basis, where the model parameters $a_i(\boldsymbol{k}), b_i(\boldsymbol{k})$ are modeled as explicit functions of the state vector $\boldsymbol{k}$.

The regression form of Eqn. 1 is parameterized in terms of the parameter vector ($\boldsymbol{\theta} = [\ a_{1,1}\ a_{1,2}\ \ldots\ a_{i,j}\ b_{1,1}\ b_{1,2}\ \ldots\ b_{i,j} \vdots \sigma_e^2(\boldsymbol{k})\ ]^T \quad \forall\ \boldsymbol{k}$) to be estimated from the measured signals via Weighted Least Squares (WLS) as it is suggested by the Gauss-Markov theorem [14].

$$J^{\text{WLS}} = \frac{1}{N}\sum_{t=1}^{N} \boldsymbol{e}^T[t]\boldsymbol{\Gamma}_{\boldsymbol{e}[t]}^{-1}\boldsymbol{e}[t] = \frac{1}{N}\boldsymbol{e}^T\boldsymbol{\Gamma}_{\boldsymbol{e}}^{-1}\boldsymbol{e}, \quad \widehat{\boldsymbol{\theta}}^{\text{WLS}} = \left[\boldsymbol{\Phi}^T\boldsymbol{\Gamma}_{\boldsymbol{e}}^{-1}\boldsymbol{\Phi}\right]^{-1}\left[\boldsymbol{\Phi}^T\boldsymbol{\Gamma}_{\boldsymbol{e}}^{-1}\boldsymbol{y}\right] \quad (2)$$

In these expressions $\boldsymbol{\Gamma}_{\boldsymbol{e}} = E\{\boldsymbol{e}\boldsymbol{e}^T\}$ ($\boldsymbol{\Gamma}_{\boldsymbol{e}} = \boldsymbol{\Gamma}_{\boldsymbol{e}[t]} \otimes \boldsymbol{I}_N$, with $\boldsymbol{I}_N$ designating the $N \times N$ unity matrix) designates the residual covariance matrix, which is estimated via Ordinary Least Squares. The final residual variance and residual covariance matrix estimates are:

$$\widehat{\sigma}_e^2(\boldsymbol{k}, \widehat{\boldsymbol{\theta}}^{\text{WLS}}) = \frac{1}{N}\sum_{t=1}^{N} e_{\boldsymbol{k}}^2[t, \widehat{\boldsymbol{\theta}}^{\text{WLS}}], \quad \widehat{\boldsymbol{\Gamma}}_{\boldsymbol{e}[t]} = \frac{1}{N}\sum_{t=1}^{N} \boldsymbol{e}[t, \widehat{\boldsymbol{\theta}}^{\text{WLS}}]\boldsymbol{e}^T[t, \widehat{\boldsymbol{\theta}}^{\text{WLS}}] \quad (3)$$

The least absolute shrinkage and selection operator (LASSO) posts an l^1-norm on the coefficient of projection ($\boldsymbol{\theta}$) that effectively reduces some entries to zero. The WLS-LASSO problem is formulated as:

$$\widehat{\boldsymbol{\theta}}^{LASSO} = \arg\min_{\boldsymbol{\theta}} \frac{1}{N}\boldsymbol{e}^T\boldsymbol{\Gamma}_{\boldsymbol{e}}^{-1}\boldsymbol{e} + \lambda||\boldsymbol{\theta}||_1 \quad (4)$$

BAYESIAN DAMAGE ESTIMATION

Having the regularized VFP-ARX as the forward model, the Monte-Carlo Markov Chain (MCMC), with Adaptive Metropolis (AM) sampling, is applied to the inverse problem. To be estimated, the damage parameter is designated with a prior distribution $\pi(\boldsymbol{k})$. Considering a residual covariance that is unknown in realistic conditions, a prior distribution is assumed for residual covariance as $\pi(\sigma_u^2)$. Thus, the joint prior distribution of damage parameter and residual covariance is formulated as:

$$\pi(\boldsymbol{k} : \sigma_u^2) = \pi(\boldsymbol{k})\pi(\sigma_u^2) \quad (5)$$

The likelihood function is defined based on the distribution of damage parameter $\pi(\boldsymbol{k})$, model prediction $\mathcal{M}(\boldsymbol{k})$, and measured response $\boldsymbol{y}$.

$$\mathcal{L}(\boldsymbol{k}, \sigma_u^2; \boldsymbol{y}) = \frac{1}{\sqrt{(2\pi\sigma_u)^N}} exp(-\frac{1}{2\sigma_u^2}(\boldsymbol{y} - \mathcal{M}(\boldsymbol{k}))^T(\boldsymbol{y} - \mathcal{M}(\boldsymbol{k}))) \quad (6)$$

With the prior distribution and likelihood function specified above, the corresponding posterior distribution is formulated as below:

$$\pi(\boldsymbol{k}, \sigma_u^2|\boldsymbol{y}) = \frac{1}{Z}\pi(\boldsymbol{k})\pi(\sigma_u^2)\mathcal{L}(\boldsymbol{k}, \sigma_u^2; \boldsymbol{y}) \quad (7)$$

Adaptive Metropolis (AM) method is applied as the sampling strategy, favoring its ergodicity and asymptotic properties [15].

RESULTS AND ANALYSIS

Evaluations and observations are based on a data set obtained experimentally via a vibration test. A wing section of composite skin-spar-rib construction is clamped cantilever and subjected to random vibration at different damage states. Damages are simulated by additional weights, $\{3, 6, 9, 12, 15, 18\}g$. And vibration responses are collected by 15 accelerometers for $64s$ at a sampling frequency of $512Hz$ [16]. Two case studies are set up for evaluating the performance of LASSO regularization and MCMC-AM damage estimation. The first case is 1-D estimation of damage size within $k_1 = \{0, 3, 6, 9, 12, 15, 18\}g$, assuming the damage position is known at $x = 4in$, $y = 0in$. A Uniform prior distribution ($k_1 \sim \mathcal{U}(0, 18)g$) is applied to search for all possible damage states. The 2-D case assumes unknown damage within $k_1 = \{0, 3, 6, 9, 12, 15, 18\}g$, and $k_2 = \{4, 10, 16, 22, 28, 34, 40\}in$ at $y = 0in$. The prior distribution is selected assuming a known previous damage state. For instance, this initiation of damage at $3g$ would be estimated from a prior of $k_1 \sim \mathcal{N}(0, 3)g$ and $k_2 \sim \mathcal{U}(4, 40)in$, which assumes no damage along the span direction but has the probability to be damaged. Once the damage is located, the prior of damage location (k_2) is no longer uniform. For example, when estimating damage at $k_1 = \{12\}g,\ k_2 = \{16\}in]$ the prior would be defined as $k_1 \sim \mathcal{N}(9, 3)g$ and $k_2 \sim \mathcal{N}(10, 6)in$.

Model Selection via LASSO

In order to evaluate the impact of regularization on the model identification process and provide insight for model structure selection, the Bayesian Information Criterion (BIC) and residual sum of squares normalized by the signal sum of squares (RSS/SSS) are used as preliminary characterizations of models' over-fitting and prediction accuracy. It can be observed from Figure. 1(a.1)(b.1) that LASSO regularization on full basis

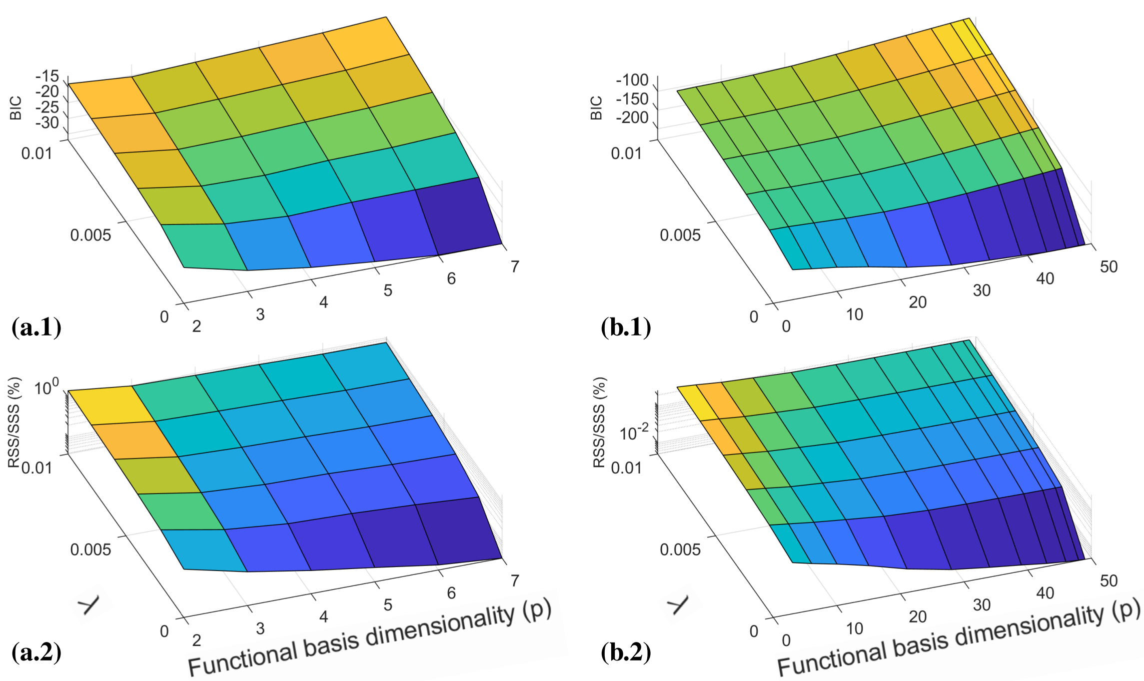

Figure 1. The evaluation of BIC and RSS/SSS for: a) FP-ARX models with basis ($p = 2 \sim 7$) and regularization parameter ($\lambda = 0 \sim 0.01$) b) VFP-ARX models with basis ($p = 2 \sim 49$) and regularization parameter ($\lambda = 0 \sim 0.01$)

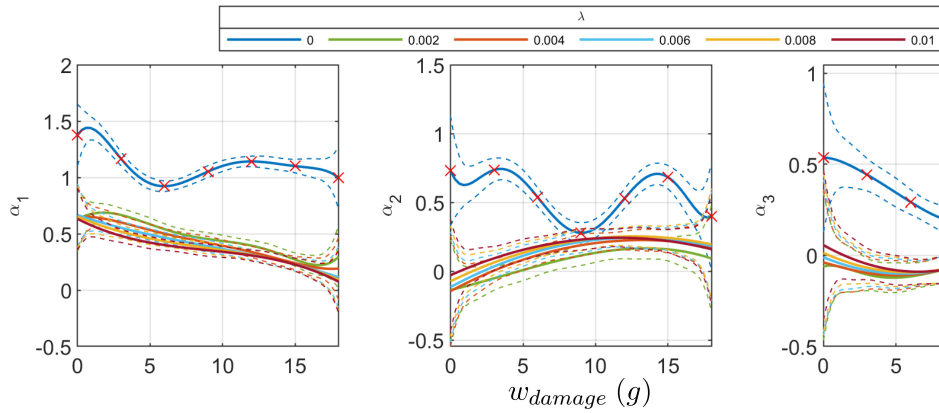

Figure 2. A comparison of the effect on ARX model parameters of FP-ARX$(54)_7$, imposed by multiple regularization levels from $\lambda = 0$(none) to $\lambda = 0.01$ with 0.002 increment. Independently estimated ARX model parameters are shown with red 'x' at each sampled damage location. 2 stds bounds for each parameter are shown by '- -'.

models, i.e. $p = 7$ and $p = 49$, can achieve similar effect to reducing the basis number (p) of the model. Thus, emphasis is placed on leveraging LASSO as a model selection technique to replace manual selection of function basis.

The effect of LASSO regularization on models' change over damage parameter space can be directly observed from arx model parameter (Eqn. 1). Fig. 2 and Fig. 3 show the comparison between unregularized and regularized models with respect to ARX parameters estimated at a single damage state. It can be easily observed that FP-ARX$(54)_7$ in Fig. 2(blue) and VFP-ARX$(54)_{49}$ in Fig. 3(a) are able to fit every independently estimated ARX parameter over the damage parameter space. However, suspicion of overfitting emerges as well. By applying regularization, the complexity of the function basis is reduced, leading to a simpler model over damage parameter space.

Finally, the effect of regularization on the process of damage parameter estimation is examined. Fig. 4 shows the damage estimation results obtained by SQP optimization via FP-ARX$(54)_7$. The initial point is set to $\boldsymbol{k}_0 = [0]g$ based on the assumption that is no initial damage. It can be observed from Fig. 4(a)(c) that SQP is able to find the local minimum. But it can also be trapped in local minimum as it is shown in Fig. 4(b). After applying regularization of $\lambda = 0.006$, the RSS functions are modified to eliminate some

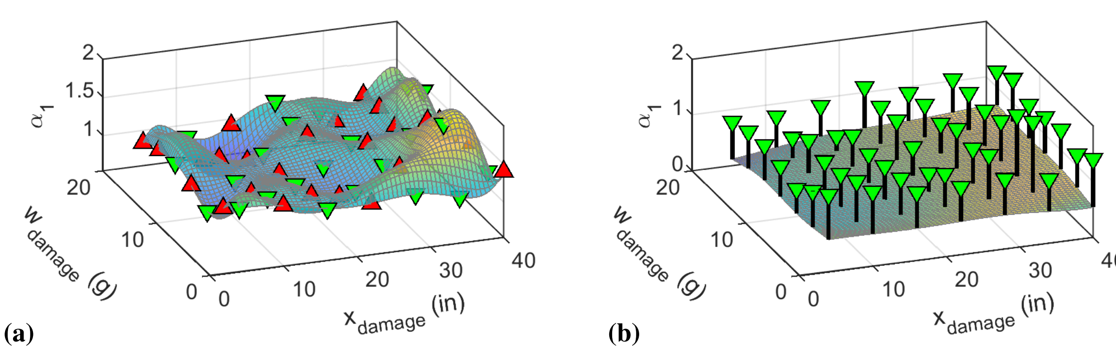

Figure 3. ARX model parameters of VFP-ARX$(54)_{49}$ with (a)$\lambda = 0$(none) and (b)$\lambda = 0.001$. Independently estimated ARX model parameters are shown with 'Δ' at each damage location, where green ones lie above the surface and the red ones vice versa.

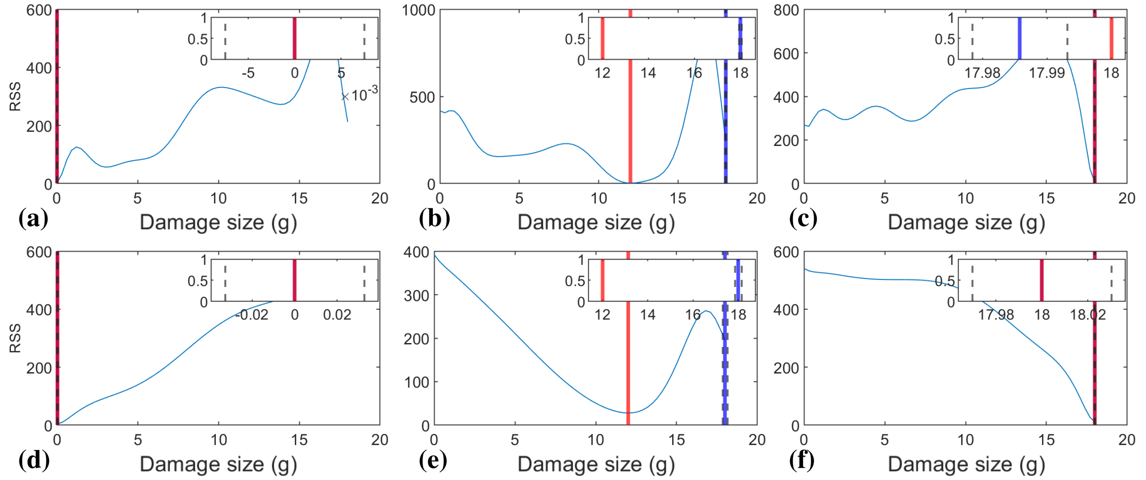

Figure 4. Representative damage state estimation results, via GA-SQP, are shown for FP-ARX$(54)_7$ at damage states: a)d) [0]g, b)e) [6]g, and c)f) [24]g. The effect of regularization is reflected by comparing: a)b)c) $\lambda = 0$, and d)e)f) $\lambda = 0.006$.

local minimums as it is shown in Fig. 4(d)(f). A similar effect of regularization is also observed in 2D damage cases that are shown in Fig. 7.

Bayesian Damage Estimation Under LASSO

In this section, the proposed Bayesian estimation approach is compared with pure SQP optimization. Unlike the SQP method that eventually converges to one point estimate and obtains probabilistic distribution from model residual series, Bayesian inversion-based method (MCMC-AM) extracts estimation from the converged posterior samples. Fig. 4 shows the damage estimation results obtained by SQP optimization via FP-ARX$(54)_7$. The initial point is set to $\boldsymbol{k}_1 = [0]g$ based on the assumption that there is no initial damage. Then, tt can be observed from Fig. 4(a)(c) that SQP is able to find the

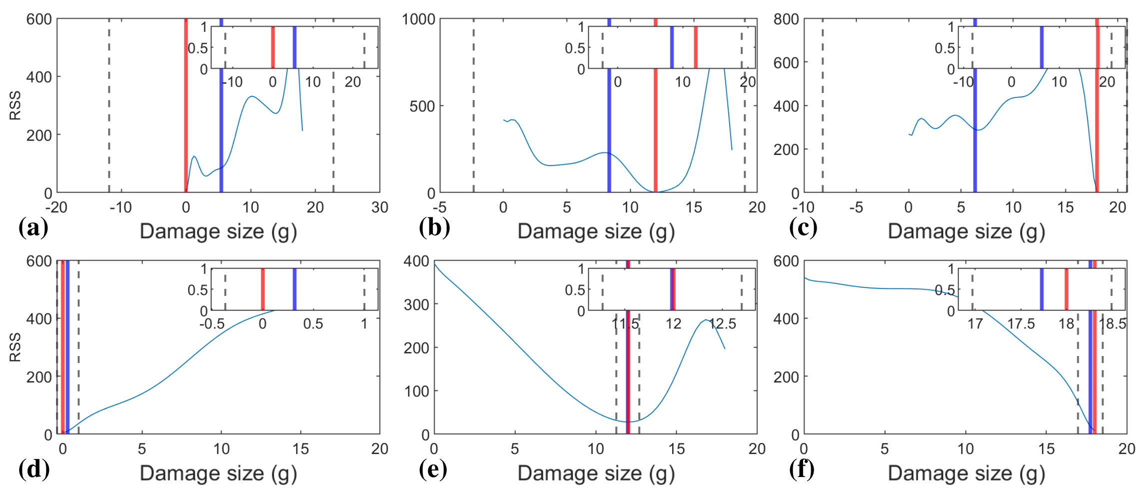

Figure 5. Representative damage state estimation results, via MCMC-AM, are shown for FP-ARX$(54)_7$ at damage states: a)d) [0]g, b)e) [6]g, and c)f) [24]g. The effect of regularization is reflected by comparing: a)b)c) $\lambda = 0$, and d)e)f) $\lambda = 0.006$.

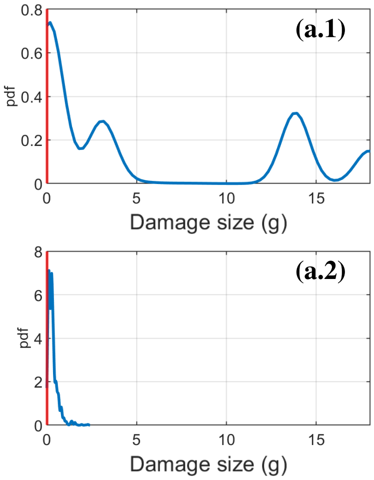

Figure 6. Posterior PDFs regarding damage size (g) are shown at three representative damage cases, where (a.1∼2) is $\boldsymbol{k} = [0]\ g$, (b.1∼2) $\boldsymbol{k} = [12]\ g$ and, (b.1∼2) $\boldsymbol{k} = [18]\ g$. The effect of regularization is shown by the comparison between (a∼c.1) $\lambda = 0$, and (a∼c.2) $\lambda = 0.006$. The true damage state is designated by red vertical line.

local minimum. But it can also be trapped in local minimum as it is shown in Fig. 4(b). After applying regularization of $\lambda = 0.006$, the RSS functions are modified to eliminate some local minimums as it is shown in Fig. 4(d)(f). A similar effect of regularization is also observed in 2D damage cases that are shown in Fig. 7. When using the regularized model, the estimations provided by the Bayesian approach are acceptable in most of the representative cases shown in Fig. 6(d)∼(f) and Fig. 9(e)∼(f). In the case shown in Fig.

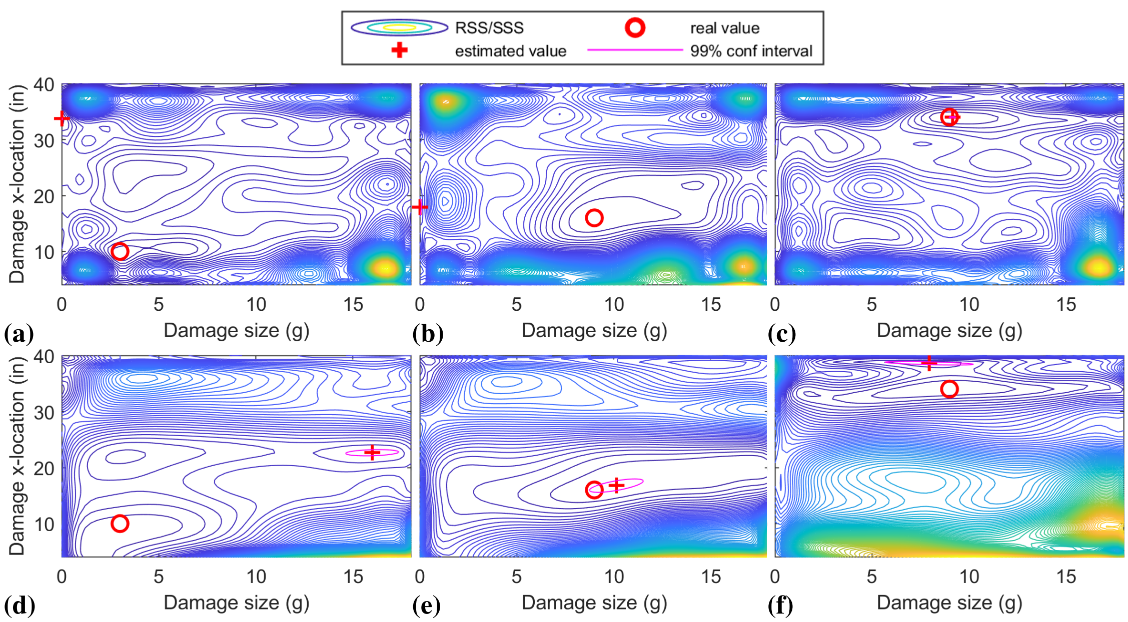

Figure 7. Representative damage state estimation results, via GA-SQP, are shown for VFP-ARX$(54)_{49}$ at damage states: a)d) $[3,\ 10]$g/in, b)e) $[9,\ 16]$g/in, and c)f) $[9,\ 34]$g/in, where the regularization is: a)b)c) $\lambda = 0$, d)e)f) $\lambda = 0.0006$, and g)h)i) $\lambda = 0.001$. The true damage state is designated by red 'o' and the damage estimation by red 'x'. The 99% $C.I.$ of damage estimation is the magenta ellipses over the colored RSS/SSS contour.

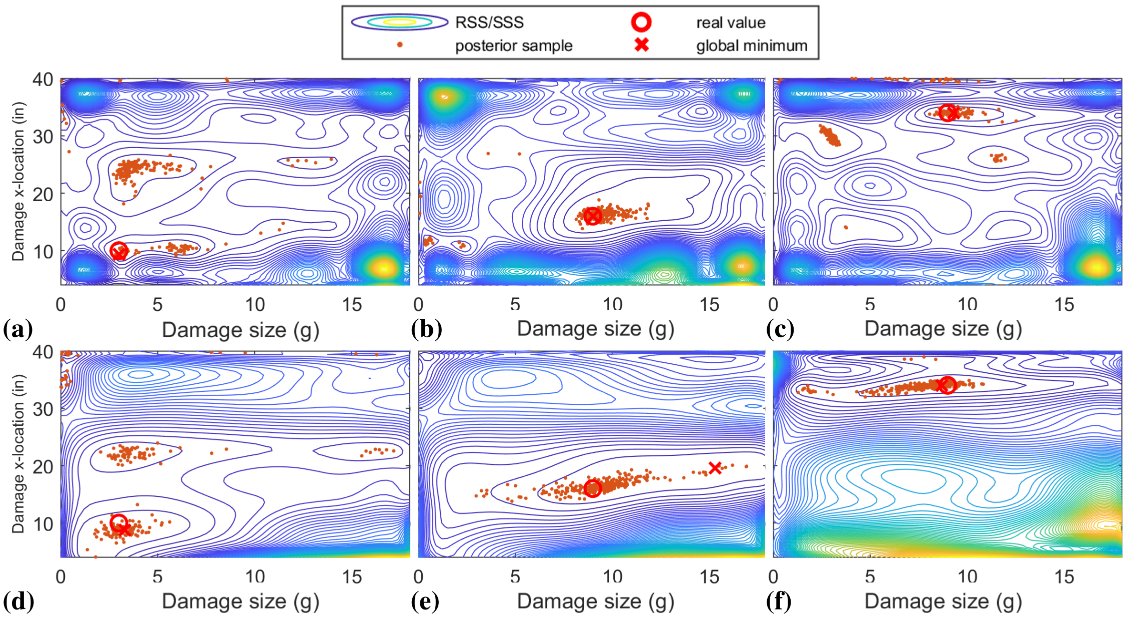

Figure 8. Representative damage state estimation results, via MCMC-AM, are shown for VFP-ARX$(54)_{49}$ at damage states: a)d)g) $[3,\ 10]$g/in, b)e)h) $[9,\ 16]$g/in, and c)f)i) $[9,\ 34]$g/in, where the regularization is: a)b)c) $\lambda = 0$, d)e)f) $\lambda = 0.0006$, and g)h)i) $\lambda = 0.001$.

9(d), a good damage state estimation can also be achieved via thresholding or advanced probabilistic models.

The effect of regularization can be observed from the comparison of the final posterior distribution between regularized and unregularized models. When the model is unregularized, a flat prior converges to multiple peaks as is shown in Fig. 6(a$\sim$c.1). Without using advanced probabilistic distribution models, the estimate of damage is ob-

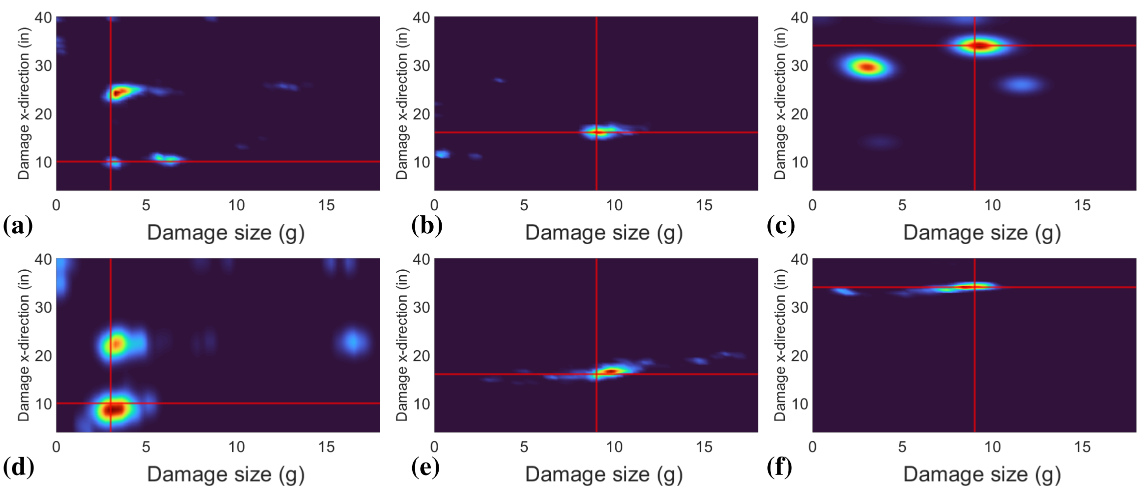

Figure 9. Representative MCMC-AM posterior distribution results are shown for VFP-ARX$(54)_{49}$ at damage states: a)d)g) $[3,\ 10]$g/in, b)e)h) $[9,\ 16]$g/in, and c)f)i) $[9,\ 34]$g/in, where the regularization is: a)b)c) $\lambda = 0$, d)e)f) $\lambda = 0.0006$, and g)h)i) $\lambda = 0.001$. The true damage state is designated by red vertical and horizontal lines.

tained from the mean value and variance/covariance of the entire posterior distribution. Thus, it can be seen in Fig. 5 that the final damage estimation deviates widely from the true damage size and the corresponding $C.I.$ covers the entire range of candidate damage sizes. Consequently, it becomes clear that this process without regularization doesn't provide useful predictions, whereas regularization provides improvement. Fig. 6(a$\sim$c.2) shows the final posterior when regularization ($\lambda = 0.006$) is applied, where the posterior distributions converge to a single peak. As a result, the damage estimation via non-parametric statistical moments can provide good results shown in Fig. 5(d$\sim$f), where the point estimates are accurate and enclose true damage value in $C.I.$.

In 2D cases, regularization forces the final posterior distribution to a single peak as it is shown in Fig. 9(b)(c)(e)(f). A different scenario is observed by comparing Fig. 9(a) and (d), where multiple peaks still exist in the posterior when regularization is applied. However, as the RSS function is smoothed in the vicinity of the true damage state, more MCMC chains converge to the true damage state as it is shown in Fig. 8(a)(d). Thus, the highest peak in the posterior distribution returns to the true damage state.

CONCLUDING REMARKS

This work aims to investigate using of LASSO regularization on the VFP-ARX model for structure selection and applying Bayesian inversion on damage state estimation instead of the deterministic optimization approach. It is shown that LASSO regularization can serve the purpose of model structure selection by operating on VFP-ARX models' coefficient of projection and introducing sparsity as necessary, thus achieving better generalizability and reducing local minimums in the RSS function. It is shown that LASSO regularization provides a better-conditioned Bayesian inversion problem.

Using the Bayesian inverse for damage estimation provides an approach to describe the stochastic nature of structural damage. Additionally, the Bayesian approach allows the incorporation of prior selection to reduce ambiguity based on existing knowledge of the damage state. Damage estimations for both damage size and location are shown to be successful for a wing structure using proposed method.

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