Piece-wise Ritz Analysis of Beams Subjected to Discontinuity in Slope

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ABSTRACT

This study presents a Ritz-type analysis for obtaining the vibration frequencies and mode shapes of beams that have a discontinuity in slope. Such structures have a wide range of applications in engineering as they often represent structural idealizations of complex structures such as robotic arms, crack modeling and foldable wings of aircraft. In the present study, the beams have been modeled using Euler Bernoulli's theory, and the discontinuity in slope is represented by a torsional spring connecting the two segments of the beam. The beams are discretized into subdomains based on the location of the discontinuity. Legendre polynomials are used to define the displacement variation over individual uniform domains. The continuity of displacement is applied at the interface of adjacent subdomains and is enforced into the global system of equations using a condensation procedure. This procedure eliminates the dependent Ritz constant obtained from the displacement continuity. This study focuses on obtaining the vibration frequencies and mode shapes of a simply-supported beam with a discontinuity in slope and compares the results with the analytical solution.

The paper would be of interest to researchers involved with the structural health monitoring of beams with cracks, robotic arms, and vibrations of folding wings like the one being considered for 777-X.

INTRODUCTION

Dang *et al.* [1] and Parthasarathy and Kapania [2] proposed a piecewise Ritz method to handle discontinuity in beams either subjected to discontinuous load or stepped beam. The beams are discretized into beams without discontinuity (subdomains) based on the location of the discontinuity. Trial functions are chosen individually for each subdomain based on their boundary conditions. Later, the compatibility conditions are applied at the interface of adjacent subdomains and are enforced into the global system of equations using a condensation procedure or the method of Lagrange multipliers. The proposed approach is a powerful tool for approximating displacements, shear force and bending

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moment. In all the previous studies, the displacement and slope were continuous at the point of discontinuity. However, what if either of these aforementioned quantities were discontinuous? Will, in that case, the current method still hold? The current study deals with one such problem, i.e., finding the natural frequencies and mode shapes of vibration of beams with discontinuity in slope. In the case of the general Ritz method, choosing a single continuous and differentiable function as trial functions would not yield accurate results.

To represent the discontinuity in slope, a torsional spring is placed along the span of the beam, i.e., each end of the torsional spring is connected to different segments of the beam, as shown in Figure 1. Such structures have a wide range of applications in the engineering field and generally represent structural idealizations of complex structures. Cracks are one of the main modes of structural failure; they represent a decrease in stiffness and natural frequency, causing structures to fail during their operational life. As a result, when the structure is subjected to an external load, the crack will tend to open and close, causing the structure to twist or bend. Thus, torsional springs can be used to model cracks [3]. In aerospace engineering, torsional springs are often used to represent the folding wing configuration because they can simulate the wing's behavior as it is folded and unfolded. In a folding wing configuration, the outer wing tip is attached to the inner wing of an aircraft via a hinge or pivot point, which allows the wing to rotate about its longitudinal axis. Torsional springs can be used to model this rotational behavior by providing a resistance to twisting or torsion, which is analogous to the resistance that the wing experiences as it is folded or unfolded [4,5].

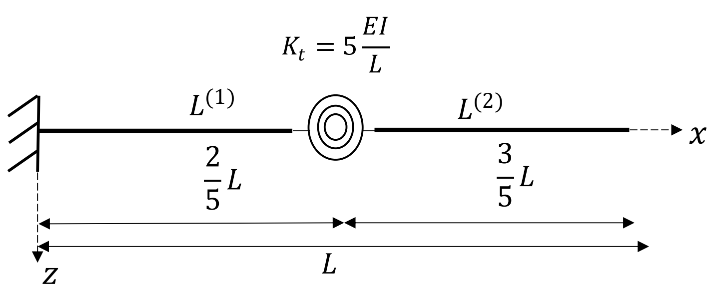

Figure 1. Euler-Bernoulli Beam

NORMAL MODES: EULER-BERNOULLI BEAM WITH DISCONTINUITY IN SLOPE

An Euler Bernoulli beam is considered with a discontinuity in slope which is represented by a torsional spring (stiffness K_t) connecting two segments of the beam as shown in Figure 1. The system's frequencies and corresponding mode shapes are of particular interest. The beam is divided into beam segments or subdomains without any discontinuities and has length $L^{(i)}$ and is represented by a non-dimensional coordinate system $\xi - \eta$. Where, $L^{(i)}$ represents the length of the i^{th} subdomain. For the system shown in Figure 1, the left beam segment is subdomain 1, and the right beam segment is subdomain 2. The transformation from Cartesian $x - y$ coordinate system to a non-dimensional coordinate system is given by:

$$x = \frac{1}{2}(L^{(i)}\xi + (x_i + x_{i+1})); \xi \in [-1, 1] \tag{1}$$

Later we obtain the strain energy and kinetic energy in the non-dimensional coordinate system for each subdomain [1, 2], which is used for finding the Lagrangian functional. The Lagrangian functional $\ell^{(i)}$ is minimized only if the unknown Ritz coefficient vector q satisfies the below Euler-Lagrangian Equation

$$\frac{d}{dt}\left(\frac{\partial \ell^{(i)}}{\partial \dot{q}^{(i)}}\right) - \frac{\partial \ell^{(i)}}{\partial q^{(i)}} = 0; m = 1...N^{(i)} \tag{2}$$

Where $N^{(i)}$ represents the number of trial functions and $q^{(i)}$ represents the vector of unknown Ritz coefficients for the i^{th} subdomain. ϕ_m are the trial functions, and for the current study, Legendre polynomials are chosen. The above equation leads to a standard eigenvalue problem

$$(K^{(i)} - \omega^2 M^{(i)})q = 0 \tag{3}$$

Where

$$\begin{aligned} K_{mn}^{(i)} &= \frac{8}{(L^{(i)^3})}\int_{-1}^{1} EI^{(i)}\phi_m^{''}(\xi^i)\phi_n^{''}(\xi^i)d\xi \\ M_{mn}^{(i)} &= \frac{L^{(i)}}{2}\int_{-1}^{1} m(\xi^{(i)})\phi_m(\xi^i)\phi_n(\xi^i)d\xi \end{aligned} \tag{4}$$

The stiffness and mass matrices for each domain are assembled as shown below.

$$K = \begin{bmatrix} K^{(1)} & 0 & 0 & ... \\ 0 & K^{(2)} & 0 & ... \\ ... & ... & ... & K^{(N_S)} \end{bmatrix}, M = \begin{bmatrix} M^{(1)} & 0 & 0 & ... \\ 0 & M^{(2)} & 0 & ... \\ ... & ... & ... & M^{(N_S)} \end{bmatrix} \tag{5}$$

Where K and M are the global stiffness and mass matrices, respectively. N_S represents the number of subdomains. For the current system, there are only two subdomains, and the torsional spring is accounted for by creating another matrix that only consists of terms related to the stiffness of the spring.

$$K_{spring} = \begin{bmatrix} k_{spring}^{(1)} & k_{spring}^{(2)} \\ k_{spring}^{(3)} & k_{spring}^{(4)} \end{bmatrix} \tag{6}$$

Where

$$
\begin{aligned}
k^{(1)}_{spring} &= K_t \frac{4}{L^{(1)^2}} \phi'^{(1)}_m(1)\phi'^{(1)}_n(1) \\
k^{(2)}_{spring} &= -K_t \frac{4}{L^{(1)}L^{(2)}} \phi'^{(1)}_m(1)\phi'^{(2)}_n(-1) \\
k^{(3)}_{spring} &= -K_t \frac{4}{L^{(1)}L^{(2)}} \phi'^{(2)}_m(-1)\phi'^{(1)}_n(1) \\
k^{(4)}_{spring} &= K_t \frac{4}{L^{(2)^2}} \phi'^{(2)}_m(-1)\phi'^{(2)}_n(-1)
\end{aligned}
\tag{7}
$$

$\phi'^{(i)}_m$ represents the trial function for the i^{th} subdomain. The above spring terms are in the transformed non-dimensional $\xi - \eta$ coordinate system. Finally, the global or total stiffness matrix of the system is given by the sum of the stiffness matrix of the beam and spring. This can be represented as follows:

$$
K = \begin{bmatrix} K^{(1)} + k^{(1)}_{spring} & k^{(2)}_{spring} \\ k^{(3)}_{spring} & K^{(2)} + k^{(4)}_{spring} \end{bmatrix} \tag{8}
$$

where $K^{(1)}$ and $K^{(2)}$ represent the stiffness matrices of subdomain 1 and subdomain 2, respectively. The global eigenvalue problem for the whole structure can be represented as:

$$
(K - \lambda\ M)q = Hq = 0 \tag{9}
$$

Where $H = K - \lambda\ M$ and $\lambda = \omega^2$. The above equation can be expressed in matrix form as:

$$
H = \begin{bmatrix} H^1 & \dots & \dots \\ \vdots & H^2 \dots & \dots \\ \dots & \dots & H^{N_S} \end{bmatrix} \tag{10}
$$

$H^{(i)}$ represents $K^{(i)} - \lambda M^{(i)}$. A set of equations are formed by applying the displacement and slope continuity conditions at the interface of the adjacent subdomains. If the slope is discontinuous, only the continuity of displacement is taken into consideration. These systems of equations can be represented in terms of a set of equations:

$$
[C]\{q\} = 0 \tag{11}
$$

Where {q} is a vector of the Ritz coefficients, and its dimensions depend on both the number of subdomains and the number of trial functions chosen for each domain. The dimensions of the C matrix is $(2(N_s - 1), N_s.N)$.

Dang $et\ al.$ [1] and Parthasarathy and Kapania [2] have provided a detailed formulation of the condensation procedure for eliminating dependent Ritz coefficient.

The condensation procedure is repeated for all the dependent degrees of freedom. The number of dependent degrees of freedom is directly related to the number of continuity conditions applied at the interface and the number of subdomains. The condensed geometric stiffness matrix $-M^*$ is obtained by taking a derivative of H^* with respect to λ. The condensed stiffness matrix is obtained as follows:

$$K^* = H^* + \lambda M^* \tag{12}$$

The condensed eigenvalue problem for finding the eigenvalues is in the form:

$$(K^* - \lambda M^*)q^* = 0 \tag{13}$$

For the current case, there would be just one interface condition, i.e., continuity of displacement, instead of two. The consequence of just one interface condition yields just one dependent Ritz coefficient at the junction of each adjacent sub-domain. For example, let us assume the system shown in Figure 1, and the displacement of each subdomain is approximated by five trial functions. The dimension of K and M before and after condensation is 10×10 and 9×9, respectively.

Modeling the systems, such as shown in Figure 1, using commercial software is not straightforward. The system has been modeled using Nastran, where beam segments are modeled as two beams without connection. By using the $CBUSH$ element that has 6DOFs, we model the torsional spring. In order to ensure continuity of displacement, except for the DOF corresponding to rotation, we need to provide very high stiffness values for the remaining DOFs which acts like a rigid link, ensuring displacement continuity between the last node of the first beam segment and the first node of the second beam segment.

NUMERICAL EXAMPLE: RITZ APPROACH FOR MODAL ANALYSIS OF BEAM WITH TORSIONAL SPRING

The current section discusses the application of the above formulation for a simply supported beam, as shown in Figure 1. The structure is split into subdomains without discontinuities. Next, the trial functions are chosen for the approximating displacement of each subdomain based on the boundary condition. For the simply supported beam problem, for the first subdomain, the trial functions are chosen such that at $\xi = -1$, displacement is zero, while for the second subdomain, such that at $\xi = 1$, displacement is zero.

A total of six trial functions were chosen for each subdomain of the system. The first four mode shapes for the simply-supported beam are shown in Figure 2 while the first four natural frequencies are shown in Table I.

The mode shapes are normalized by the modal displacement at $x = \frac{2L}{5}$. The plots and tables show that the results from the present method are in very good agreement with

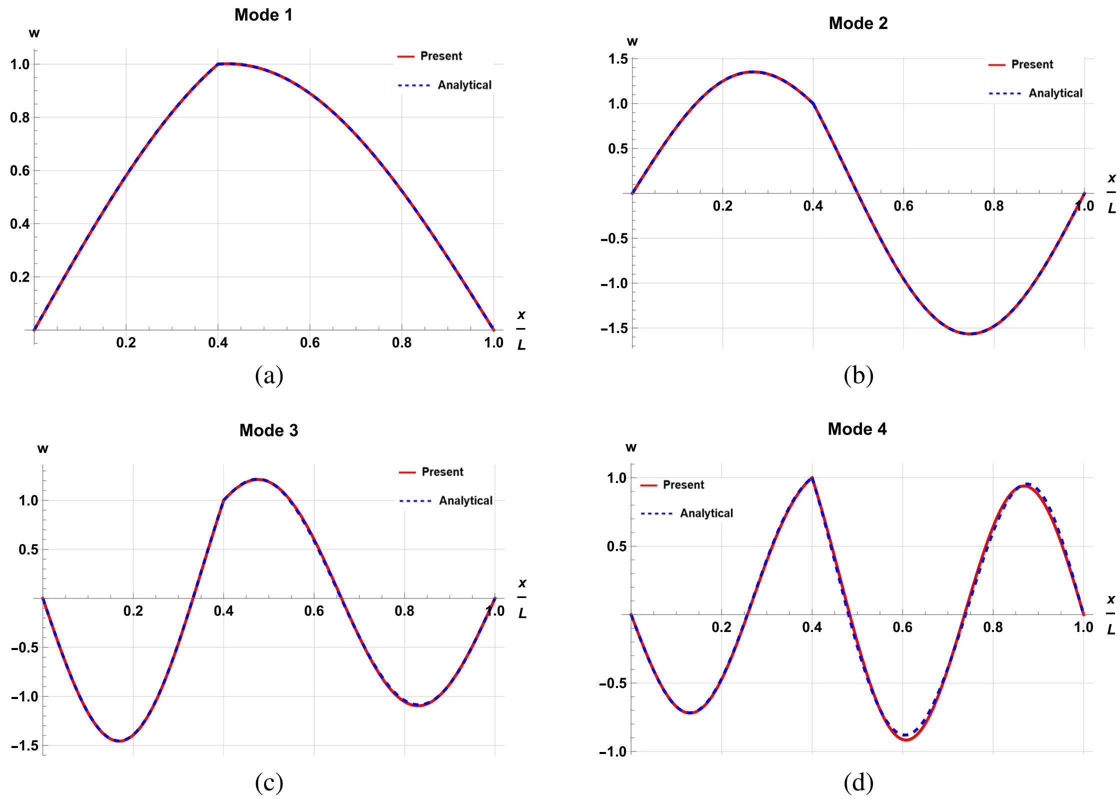

Figure 2. First Four Mode Shapes of a Simply Supported Beam with Torsional Spring located at $x = \frac{2L}{5}$

the analytical solution and the results obtained from FEM by choosing ten elements for both subdomains. As stated earlier, the effect of torsional spring causes a discontinuity in slope, which can be observed in the mode shapes. There is a small discrepancy in the fourth mode and the corresponding frequency; choosing a higher number of trial function terms would reduce the discrepancy further.

TABLE I. Comparison of First Four Natural Frequencies with Analytical Solution and FEM for a Simply Supported Beam with Torsional Spring located at $x = \frac{2L}{5}$

Mode	Analytical	FEM	Present
1	1.5779	1.5779	1.5779
2	7.0189	7.0199	7.0189
3	15.8093	15.8193	15.8174
4	26.6744	26.7259	26.8724

CONCLUDING REMARKS

The present study provides an extended version of the Ritz method, i.e., the piecewise-Ritz method for analysis related to the free-vibration of beams with discontinuity in slope. The beam is split into subdomains with no discontinuities, and a constraint matrix is formed based on displacement and slope continuity at the junction of two adjacent subdomains. The constraint equations or the interface conditions are included for removing dependent Ritz coefficients. Thus, a condensed stiffness and geometric stiffness matrix are obtained. The numerical example for a beam with discontinuity in slope that was represented by a torsional spring connecting the two segments of the beam was implemented using the proposed method. For the above case, the condensation process was implemented to eliminate dependent Ritz coefficients but with just continuity of displacement at the sub-domains interface. Six trial functions were chosen for each subdomain for numerical examples shown; the natural frequencies and mode shapes were in very good agreement with the analytical solution.

This work can be further extended by choosing different trial functions and structures being analyzed. Choosing beam characteristic functions (mode shapes of individual beams) as trial functions instead of orthogonal polynomials for vibrations of stepped beams as the latter is disadvantageous for high-frequency analysis [6]. The present study can also be extended for the analysis of plates with stiffeners. The stiffeners can be placed arbitrarily on the plate in contrast to FEM, where the nodes have to be shared to apply compatibility conditions. By independently interpolating stiffener DOFs and plate DOFs, we can use 1D trial functions for stiffener, making the analysis computationally efficient by reducing the order of integration. Since the independent trial functions (polynomials) for the stiffener would be defined in the local domain of the stiffener. Strain energy can be derived easily without any local transformations. Only one transformation would be needed at the end to add the strain energies together, which can be implemented with the help of compatibility conditions [7].

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