A Robust Pre-processing Approach of Array Samples with Hankel Matrix Completion

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Abstract. In this paper, we study the problem of recovering array samples when a few array samples are unavailable, and provide a robust pre-processing approach of recovering unavailable array samples with Hankel matrix completion. First, Alternating Direction Methods (ADM) for solving nuclear norm minimization problem with Hankel structure was discussed. Second, stability of Hankel matrix reconstruction is investigated in view of l*-Constrained Minimal Singular Value (l*-CMSV). Finally, simulation experiments illustrate that the proposed method's are robust for complex Gaussian noise.

Introduction

Estimating the directions-of-arrival (DOA) of multiple signal sources has recently attracted much renewed interest. There are many classification algorithms. One popular subspace based method is MUSIC (Multiple Signal Classification) [1,2]. Because the conventional MUSIC algorithm involves a computationally demanding spectral search step, root-MUSIC is proposed in the paper [3] in order to reduce the computational complexity of spectral MUSIC. Another popular search-free DOA estimation method is the ESPRIT (estimation of signal parameters via rotational invariance techniques) algorithm [4,5]. It exhibits lower computation and storage requirements than MUSIC algorithm by using a displacement invariant array. Recently, compressed sensing (CS) theory gained great attention from various areas in applied sciences, many novel algorithms were proposed based on CS theory, such as a semiparametric sparse iterative covariance-based estimation method (SPICE) [6]. POFEA-IHT algorithm to recovery multiple frequencies from partial phase-only data was researched in papers [7,8].

However, in many applications various array anomalies such as array element position error, sensor coupling, and array channel inconsistent, cause the actual array manifold to differ, often times significantly, from the theoretical array manifold. It will serious influence the performance of those high resolution algorithm. In this paper, we address the problem in which a few samples are unavailable. We recovery the invalid data based on Hankel matrix completion, and present the performance of this method.

The rest of paper is organized as follows. In Section II, Basic problem formulations and Primal Alternating Direction Methods of Hankel matrix completion are provided. In Section III, we study stability of Hankel matrix reconstruction in view of l*-CMSV. We further provide numerical results in Section IV and conclude the paper in Section V.

Basic Problem Formulations and ADM

In this section, we give basic problem formulations and present ADM for solving nuclear norm minimization of matrices with Hankel structure.

Consider a K sources scenario. The data set measured at time t by the arrays is [9]

\[ y(t) = As(t) + cn(t) \] (1)
where $A$ is the steering vector of the $k$-th source, $s(t)$ is the $k$-th signal, and $n(t)$ is the normal complex Gaussian-distributed noise.

Primal Alternating Direction Methods: Assume $y \in \mathbb{R}^{r+j-1}$ is a vector and $H$ is Hankel operator. If $y \in \mathbb{R}^{r+j-1}$ is obtained from model (1), it’s obvious that $\text{Rank} \ (H(y))=K$ with $n(t)=0$. Let $P_\omega$ be the orthogonal projector onto the span of vectors vanishing outside of $\omega$ so that the $(i,j)$-th component of $P_\omega(x)$ is equal to $x_{ij}$ if $x_{ij} \in \omega$ and zero otherwise. Our problem may be expressed as

$$\text{minimize} \quad \text{Rank}(H(x)) \quad \text{subject to} \quad \|y(t) - P_\omega(x)\| \leq \varepsilon$$

Rank minimization is NP-hard in general, and a popular convex heuristic for it minimizes the nuclear norm of the matrix instead of its rank [10]. The regularized version of this problem is

$$\text{minimize} \quad x \quad \text{subject to} \quad y(t) - P_\omega(x) + \mu H(x) \leq \varepsilon$$

Using the substitutions $Y = -H(x)$ and $z = y(t) - P_\omega(x)$, problem (3) can be reformulated as

$$\text{minimize} \quad p(Y, z) = \frac{1}{2} \|z\|^2 + \mu \|y\| \quad \text{subject to} \quad Y + H(x) = 0, z + P_\omega(x) = y$$

To apply the alternating direction method (ADM) for solving (4), the augmented Lagrangian function is

$$L_\beta(Y, z, x, \gamma, \Lambda) = \frac{1}{2} \|z\|^2 + \mu \|y\| - \langle \Lambda, Y + P_\omega(x) \rangle - \langle \gamma, z + P_\omega(x) - y \rangle$$

$$+ \frac{\beta}{2} \|Y + H(x)\|_F^2 + \frac{\beta}{2} \|z + P_\omega(x) - y\|_F^2$$

for each $\beta > 0$.

ADM [11,12] strategy is to first minimize $Y, z$ with respect to $(x, \gamma, \Lambda)$, minimize $x$ with respect to $(Y, z, \gamma, \Lambda)$, and then finally update the Lagrange multiplier $(\gamma, \Lambda)$.

A1. iterations of $Y$

$$Y^{k+1} = \arg \min_Y L_\beta(Y, z^k, x^k, \gamma^k, \Lambda^k) = \arg \min_Y \left\{ \mu \|y\| + \frac{\beta}{2} \|Y + H(x^k) - \frac{1}{\beta} \Lambda^k \|_F^2 \right\} = UD_z(\Sigma)V^T$$

where $H(x^k) - \frac{1}{\beta} \Lambda^k = -USV^T$, $D_z(\Sigma)$ is the singular value thresholding operator [13].

A2. iterations of $z$

$$z^{k+1} = \arg \min_z L_\beta(Y^{k+1}, z, x^k, \gamma^k, \Lambda^k) = \arg \min_z \left\{ \frac{1}{2} \|z\|^2 - \langle \gamma^k, z \rangle + \frac{\beta}{2} \|z + P_\omega(x^k) - y\|_F^2 \right\}$$

let $p_\gamma(z) = \frac{1}{2} \|z\|^2 - \langle \gamma^k, z \rangle + \frac{\beta}{2} \|z + P_\omega(x^k) - y\|_F^2$, gradient of the functional $p_\gamma(z)$ is

$$\frac{dp_\gamma}{dz} = z - \gamma^k + \beta(z + P_\omega(x^k) - y)$$

Assume $\frac{dp_\gamma}{dz} = 0$, we can obtain $z^{k+1} = \frac{\gamma^k - \beta P_\omega(x^k) + \beta y}{1 + \beta}$. 

213
A3. iterations of $x$ minimizing $L_{\beta}$ with respect to $x$ does not usually have a simple closed form solution due to the complicated quadratic terms. One way to resolve this is to add a proximal term with norm induced by a suitable positive (semi-)definite matrix to cancel out the complicated parts. Iterations of $x$ is

$$x^{k+1} = \arg \min_x \left\{ L_{\beta}(Y^{k+1}, z^{k+1}, x, y^k, \Lambda^k) + \frac{\beta}{2} \|x - x^k\|_0^2 \right\}$$

where $\|x\|_Q = x^T Q x$, $Q = \frac{1}{\sigma} I - (H^* H + P_o P_o^T)$, $\sigma < \frac{1}{r + \sigma_{\text{max}}(P_o)}$. Similarly as $z$, we can obtain

$$x^{k+1} = x^k - \frac{\sigma}{\beta} \left( H^* \Lambda^k + P_o^* y^k \right) + \sigma H^* (y^{k+1})$$

A4. update the Lagrange multiplier ($\Lambda, \gamma$)

$$\gamma^{k+1} = \gamma^k - \beta \left( z^{k+1} + P_o(x^{k+1}) - y \right)$$

$$\Lambda^{k+1} = \Lambda^k - \beta \left( y^{k+1} + H(x^{k+1}) \right)$$

The convergence of these algorithms has been well-studied (see e.g. [12,14] and the many references therein).

**ROBUST Analysis of PADM**

In many applications the observed measurements are always corrupted by different kinds of noise which may affect every entries of the data matrix. In order to further complete the theory of PADM, it’s necessary to research the stability of low-rank hankel matrix reconstruction with respect to noise. We first provide some important definitions which will be used throughout this section.

**Definition:** matrix LASSO optimization problem. The mLASSO solves the following optimization problem

$$\min_{Z \in \mathbb{R}^{n \times 2}} \frac{1}{2} \| y - A(Z) \|_2^2 + \mu \| Z \|_F.$$  

Note that the operator $P_o$ can be decomposed into $B \circ H$, then it follows from (3) that

$$\min_{Z \in \mathbb{R}^{r^2 \times r}} \frac{1}{2} \| y - B \circ H(Z) \|_2^2 + \mu \| H(Z) \|_F.$$ 

Let $X = H(Z)$, the optimization problem (3) can regard as a special case of a more general class of mLASSO.

Cand’s and Plan show that matrix restricted isometry property (mRIP) can guarantee the stability of mLASSO[15]. However, 1-CMSV has several advantages over the mRIP in stability analysis of low-rank matrix analysis. Such as the error bounds involving 1-CMSV have more transparent relationships with the signal-to-noise ratio, and the derivation of the 1-CMSV bounds is less complicated and the resulting bounds have more concise forms [16]. Therefore we will discuss the stability of low-rank hankel matrix reconstruction in view of 1-CMSV.

**Definition:** 1-Rank. Assume $x \in \mathbb{R}^{r^2 \times 1}$ is a nonzero vector. The 1-Rank is defined as

$$\tau(x) = \frac{\|H(x)\|_F}{\|H(x)\|_2}.$$  

214
Definition: $l^*$-CMSV. For any $\tau \in [1, \eta_i]$ and any linear operator $A : \mathbb{R}^{n \times n} \mapsto \mathbb{R}^m$, define the $l^*$-CMSV of $A$ by

$$
\rho_{l^*}(A) = \inf_{x \in \mathbb{R}^{n \times n}, \tau(x) \leq \tau} \frac{\|A(H(x))\|_F}{\|H(x)\|_F}.
$$

(15)

The function $\tau(x)$ is indeed a measure of the effective rank, i.e., suppose $\text{Rank}(H(x)) = r$; then, Cauchy-Schwarz inequality implies that $\tau(x) \leq r$. Similar to the proof of Proposition 1 in the paper [16], we can obtain

Proposition: If $x \in \mathbb{R}^{i+j-1}$ with $\tau(x) = r$ and the noise $n$ satisfies $\|B(n)\| \leq \kappa \mu$, the error vector $h = \bar{x} - x$ satisfies

$$
\tau(h) \leq cr
$$

(16)

where $c = 8/(1-\kappa)^2$.

As pointed out in the paper [16], probability analysis shows that at least for isotropic and subgaussian operators, the $l^*$-rank characterization is as good as the null space property characterization [17]. Similar to the proof of Theorem 1 in the paper [16], we have

Theorem 1: Under the assumption of Proposition 1, the error vector $h = \bar{x} - x$ satisfies

$$
\|H(h)\|_F \leq \frac{1+\kappa}{1-\kappa} \frac{2\sqrt{2r}}{\rho_{8r}(B)} \mu
$$

(17)

It's obvious that $\|h\| \leq \|H(h)\|_F$. Combing with (3.17), we have

Theorem 2: Under the assumption of Proposition 1, the error vector $h = \bar{x} - x$ satisfies

$$
\|\bar{x} - x\| = \|h\| \leq \frac{1+\kappa}{1-\kappa} \frac{2\sqrt{2r}}{\rho_{8r}(B)} \mu
$$

(18)

Numerical Experiments of PADM

In this section, numerical experiments with varieties of the value of parameter $\alpha$, parameter $\mu$ are given. For each setting of parameters, we show the average errors over 600 trials. Our implementation was realized with MATLAB. Without loss of generality, we assume that $i=j=6$. Fig.1 shows result in a trail intuitively by PADM.

Figure 1. Show the variance of recovered low-rank Hankel matrix.
For each $\alpha$, we repeat experiment 600 times. Fig. 1 shows the variance of recovered low-rank Hankel matrix by PADM under different parameter $\alpha$ with 1-2 samples are unavailable respectively. The y-axis corresponds to variance of recovered low-rank Hankel matrix. The x-axis corresponds to parameter $\alpha$ varying from 0.01 to 0.05. The phenomenon of Fig. 1 verifies stability of Hankel matrix reconstruction by number experiments with PADM.

**Numerical Experiments of PADM**

In this paper, we address the problem of recovering array samples when a few array samples are unavailable, and also we have investigated the stability of PADM for solving nuclear norm minimization optimization problem with Hankel structure in view of $l_1$-Constrained Minimal Singular Value ($l_1$-CMSV). Numerical results verifies stability of Hankel matrix reconstruction to complex Gaussian-distributed noise.

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**References**


