Study of Bank’s Optimal Loan-to-value Ratios Impacted by Liquidation Delay and Liquidity Risk in Environment with Uncertainty

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Abstract. We consider the inventory financing commodities such as copper and aluminum with short-term price following geometric Brownian motion. We take the research of the small and medium enterprises’ default probability impacted by such factors as the price volatility, the liquidation delay and the liquidity risk of the collateral. We establish the maximum and minimum levels of risk tolerance as the bank's maximum and minimum risk preferences in environment with Knightian uncertainty. We build the models about the loan-to-value ratios with the bank's maximum and minimum risk preferences. And we give the explicit solutions of models about the LTV ratios. Finally, we get the numerical solutions of models.

Introduction

The analysis of the loan-to-value ratios about inventory financing belongs to the research areas about the influence of collateral on credit risk exposure and the determination of collateral’s the loan-to-value ratios. Some scholars have made research in this area from two different ideas. Some people follow the structured method of Merton [1]. Stulz&Johnson [2] have studied the influence of collateral on the pledge secured debt pricing. Jokivuolle & Peura [3] have followed this structured thinking to study collateral’s loan-to-value ratios. Then, Cossin & Hricko [4] have determined the discount rate h of the collateral (assume the LTV ratio ω , then 1-h=ω ). Cossin & Huang [5] have given the exogenous probability of corporate defaults, followed the simplified ideas proposed by Jarrow& Turnbul [6], and got a discount rate of the collateral which is consistent with bank’s risk tolerance. Wang et al. [7] study the optimal operational strategies in preorder financing model of a supply chain under the bank’s risk limit. Li Yixue and others [8] have studied the discount rate of the collateral for copper, aluminum and other similar transactions in the futures market.

In this paper, we still study copper, aluminum and other similar standard commodities traded in the futures market. The difference between this paper and above references is that we not only study the risks in the financial market, but also study the Knightian uncertainty. Because Knightian uncertainty exists, in other words, there is a set of many probability measures in the financial markets, the economic agents are difficult to choose which probability measure to measure a variety of situations which may arise in the future. Thus we use the set of many probability measures to build the maximum and minimum levels of risk tolerance as the bank's maximum and minimum risk preferences in environment with Knightian uncertainty. We assume that the short-term prices of inventory financing commodities follow geometric Brownian motion. We study the firms’ default probability impacted by the price volatility and the liquidity risk of the collateral. The models of the loan-to-value ratios are established with the bank's maximum and minimum risk preferences. The numerical solutions of models about the loan-to-value ratios are calculated. Finally, we analyze the impacting factors such as Knightian ambiguity parameter and liquidity risk on the models.
Models

Model Assumptions

Let \((\Omega, F, P)\) be a complete probability space. The price processes of the stocks of standard commodities traded in the futures market satisfy the following SDE, where the parameters of \(r, \sigma, s\) are constants respectively, and the process \(\{W_t^Q\}_{0 \leq t \leq T}\) is a Brownian motion with the risk-neutral probability measure \(Q\). \(T\) is contractual maturity.

\[
dS_t = S_t (r dt + \sigma dW_t^Q), \quad S_0 = s, 0 \leq t \leq T
\]

We assume that the financing pledge goods are standard commodities such as aluminum and copper. At the initial time, the Borrower will give \(a_0\) units the stock of goods with \(S_0\) price to warehousing enterprises with legal qualifications in order to apply for a loan to the bank. The bank will give a \(\omega\) ratio of loan amount for each unit of the collateral. When the bank commissions the logistics and warehousing companies to save the pledge of stocks of goods, there will be a cost during the loan period. The bank will hold the costs credited to the loan interest rate, and finally charged to the borrower enterprise unified.

Then the assumptions of inventory financing model are listed below.

1. We assume the loan interest rate is a constant \(R\). The loan principal and interest of the subject at time \(t\) is \(v_t = v_0 e^{Rt}\).
2. We assume that the frequency of covering short positions is \(M\), the time interval of covering short positions is \(\tau (\tau \cdot M = T)\).
3. We assume the probability of default about the loans is \(Q_0\). It is exogenously given and calculated for the year.
4. At the time to cover short positions, as long as the loanable value of the pledge of stock of goods deviates from the loan principal and interest, there should be a margin call in order to restore balance.

At the beginning of the \(m\) \((m = 1, 2, \ldots, M)\) period, the principal and interest of the loan is \(v_0 e^{R(m-1)}\). Now the inventory number is \(a_{m-1}\), and the market value of every unit is \(S_{m-1}\). Thus \(v_0 e^{R(m-1)} = \alpha a_{m-1} S_{m-1}\). At the end of the \(m\) period, one of the following three cases will appear.

1. When \(v_0 e^{R_{tm}} = \alpha a_{m-1} S_{m-1}\), the Borrower neither extracts nor adds pledged inventories, the contract continues.
2. When \(v_0 e^{R_{tm}} < \alpha a_{m-1} S_{m-1}\), the Borrower can extract some pledged inventories so as to \(v_0 e^{R_{tm}} = \alpha a_{m-1} S_{m-1}\), the contract continues.
3. When \(v_0 e^{R_{tm}} > \alpha a_{m-1} S_{m-1}\), the Borrower must add some pledged inventories so as to \(v_0 e^{R_{tm}} = \alpha a_{m-1} S_{m-1}\), the contract continues. Otherwise, if the Borrower defaults, the contract will be terminated into liquidation, the bank will suffer the biggest loss \(v_0 e^{R_{tm}} - a_{m-1} S_m\).

In this paper, we study the case that the borrower corporate defaults. Then the banks’ liquidation delay and liquidity risk impact the loan- to- value ratios. The banks often seek the statutory procedures, bargaining or finding a buyer which delay the time when the pledged stocks of goods is realized into cash. We assume the period of the liquidation delay is \(\delta \tau\). When the borrower corporate defaults, the banks’ loss will not be determined based on the value of pledged stocks of goods at \(m \tau\) moment, but based on the value of pledged stock of goods at \((m + \delta) \tau\) moment. In addition, when the banks clear the pledged stocks of goods, they may encounter liquidity risk of the stocks of goods. That is to say that the pledged stocks of goods can’t be realized the full price due to the liquidation of inventory goods. The market value of pledged stocks of goods must be discounted. This paper simply assumes that the loss caused by liquidity risk is the ratio \(\rho\) of market value.
At the beginning of the $m$ period, $v_0 e^{R t(m-1)} = \omega a_{m-1} S_{m-1}$, thus $a_{m-1} = \frac{v_0 e^{R t(m-1)}}{\omega S_{m-1}}$. At the moment of $(m + \delta) \tau$, the possible loss of banks is in the following.

$$loss_m = e^{r(m + \delta)\tau} (v_0 e^{R t(m+\delta)\tau} - a_{m-1} S_{m+\delta} (1 - \rho)) = e^{r(m + \delta)\tau} (v_0 e^{R t(m+\delta)\tau} - \frac{v_0 e^{R t(m-1)}}{\omega} S_{m+\delta} (1 - \rho))$$

(2)

**Maximum and Minimum Risk Preference**

In order to consider the financial market with Knightian uncertainty, we introduce a feasibly controllable set $\Theta = (\theta)_{0 \leq t \leq T}$, where the constant $k$ is nonnegative. Chen Zengjing and Larry Epstein [9] call $\Theta - k$ – ignorance.

The set $\Pi^\theta$ of equivalent probability measures is constituted from the set $\Theta$.

$$\Pi^\theta = \{ Q^\theta \mid \frac{dQ^\theta}{dQ} \bigg|_{\mathcal{F}_t} = \exp\{ -\theta W_t^Q - \frac{1}{2} \theta^2 T \}, (\theta)_{0 \leq t \leq T} \in \Theta \}$$

(3)

The Knightian uncertainty of the financial market is characterized by the set $\Pi^\theta$. Because the banks do not know which probability measure of $\Pi^\theta$ should be used to calculate the probability of loss. From the conservative point of view, they will calculate the maximum and minimum probabilities of the event that the loss of banks is greater than the maximum loss level. That is to say, for any measurable event $A$, define

$$P_{\max}(A) = \max_{\theta \in \Pi^\theta} \{ Q^\theta(A) \}$$

(4)

$$P_{\min}(A) = \min_{\theta \in \Pi^\theta} \{ Q^\theta(A) \}$$

(5)

We build the models (4) and (5) as the maximum and minimum risk preferences of banks so as to calculate the loan-to-value ratios of standard inventory financing.

**Determining LTV Ratios**

Let $L$ be the maximum loss level that the bank is willing to bear, and $L$ be the function of the underlying asset. In convenience, we let $L = lv_0$. Then we can calculate the probabilities of the event that $loss_m$ is greater than $L = lv_0$ with the maximum and minimum risk preferences of banks.

**Theorem2.1** If we assume the period of the liquidation delay is $\delta \tau$ and the liquidity risk is the ratio $\rho$ of the collateral market value, we can build models as follows.

$$P_{\max}(loss_m \geq L) = N(\frac{G(\omega) - (r - k\sigma - \frac{1}{2} \sigma^2)(1 + \delta)\tau}{\sigma \sqrt{(1 + \delta)\tau}})$$

(6)

$$P_{\min}(loss_m \geq L) = N(\frac{G(\omega) - (r + k\sigma - \frac{1}{2} \sigma^2)(1 + \delta)\tau}{\sigma \sqrt{(1 + \delta)\tau}})$$

(7)

where $G(\omega) = \ln(\omega e^{R t(m+\delta)\tau} - le^{R t(m-1)\tau - R t(m-1)\tau - R t(m-1)\tau})$. 350
Proof. From (1) and (3), we can get
\[
Q^2 \{\text{loss}_m \geq L \} = Q^2 \{ e^{-\rho (m+\delta)\tau} (v_0 e^{R(m+\delta)\tau} - \frac{v_0 e^{R(m-1)\tau}}{\omega} \frac{S_{m+\delta}}{S_{m-1}} (1 - \rho)) \geq L \}
\]
= \( \frac{Q^2}{2} \left\{ \frac{\omega (e^{R(1+\delta)\tau} - le^{R(m+\delta)\tau - R(m-1)\tau})}{1 - \rho} \right\} = \int_{-\infty}^{\infty} \frac{\omega (e^{R(1+\delta)\tau} - le^{R(m+\delta)\tau - R(m-1)\tau})}{1 - \rho} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx \)
= \( N(\frac{G(\omega) - (r + \sigma \theta - \frac{1}{2} \sigma^2)(1 + \delta)\tau}{\sigma \sqrt{(1 + \delta)\tau}}) \) where \( G(\omega) = \ln \frac{\omega (e^{R(1+\delta)\tau} - le^{R(m+\delta)\tau - R(m-1)\tau})}{1 - \rho} \).

Because
\[
\frac{\partial N(\frac{G(\omega) - (r + \sigma \theta - \frac{1}{2} \sigma^2)(1 + \delta)\tau}{\sigma \sqrt{(1 + \delta)\tau}})}{\partial \theta} = -\frac{1}{\sqrt{(1 + \delta)\tau}} n\left(\frac{G(\omega) - (r + \sigma \theta - \frac{1}{2} \sigma^2)(1 + \delta)\tau}{\sigma \sqrt{(1 + \delta)\tau}}\right) < 0
\]
where \( n(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \). Thus \( P_{\max}(\text{loss}_m \geq L) = N\left(\frac{G(\omega) - (r - k \sigma - \frac{1}{2} \sigma^2)(1 + \delta)\tau}{\sigma \sqrt{(1 + \delta)\tau}}\right) \).

Similarly we have \( P_{\min}(\text{loss}_m \geq L) = N\left(\frac{G(\omega) - (r + k \sigma - \frac{1}{2} \sigma^2)(1 + \delta)\tau}{\sigma \sqrt{(1 + \delta)\tau}}\right) \).

The proof is obviously completed.

Theorem 2.2 During the entire loan period, the probabilities that the bank’s losses are greater than or equal to \( L \) are then estimated as follows:

\[
P_{\max}(\text{loss} \geq L) = \sum_{m=1}^{M} (1 - \tau Q_0)^{m-1} \tau Q_0 \cdot N\left(\frac{G(\omega) - (r - k \sigma - \frac{1}{2} \sigma^2)(1 + \delta)\tau}{\sigma \sqrt{(1 + \delta)\tau}}\right)
\]
(8)

\[
P_{\min}(\text{loss} \geq L) = \sum_{m=1}^{M} (1 - \tau Q_0)^{m-1} \tau Q_0 \cdot N\left(\frac{G(\omega) - (r + k \sigma - \frac{1}{2} \sigma^2)(1 + \delta)\tau}{\sigma \sqrt{(1 + \delta)\tau}}\right)
\]
(9)

Proof. Because the probability of loan default is exogenous and independent of the particular loan, the banks are concerned with the probability of two events occurring simultaneously. One event is \( \{\text{loss}_m \geq L\} \), and the other event is the borrower defaults. We assume that the loan default probability per year is \( Q_0 \) with a uniform distribution. Therefore, the probability of the loan default in the \( m \)-th period is simply \( \tau Q_0 \). We then get that the joint probabilities of these two events under the bank’s maximum and minimum risk preference are \( \tau Q_0 P_{\max}(\text{loss}_m \geq L) \) and \( \tau Q_0 P_{\min}(\text{loss}_m \geq L) \).

Because the borrower can only default once and the probability that the borrower does not default before the \( m \) period is \( (1 - \tau Q_0)^{m-1} \). Thus, with the bank’s maximum and minimum risk preference, the probabilities that the bank losses are greater than or equal to \( L \) in the \( m \)-th period are estimated as \( (1 - \tau Q_0)^{m-1} \tau Q_0 P_{\max}(\text{loss}_m \geq L) \) and \( (1 - \tau Q_0)^{m-1} \tau Q_0 P_{\min}(\text{loss}_m \geq L) \) respectively.

The proof is obviously completed.
A Numerical Analysis

We assume that the collateral is copper in the futures market, and its price equation is driven by a Brownian motion, with \( \sigma = 0.4118 \), \( \mu = 0.255 \), \( P = 0.1358 \). We assume the following parameters in the numerical analysis: \( Q_0 = 0.9 \) (loan default probability per year), \( r = 0.03 \) (interest rate), \( R = 0.15 \) (loan interest rate), \( M = 90 \) (mark to market frequency), \( T = 90 \) days (loan time), \( l = 0.3 \) (degree of loan losses), \( k \) (Knightian uncertainty parameter) is in \( (0,1) \), \( \delta = 0 \) (liquidation delay parameter), \( \rho \) (discount ratio caused by the liquidity risk) is in \( (0,0.2) \).

Based on the formula \((8)\), we get \( \omega = 0.01 \). Based on the formula \((9)\), we get \( \omega = 0.42 \). After considering liquidation delay and liquidity risk with the Knightian uncertainty, we get the optimal LTV ratio with an interval of \((0.01,0.42)\). In the Knightian uncertainty-neutral environment, the LTV ratio \( \omega \) is 0.02, which falls into this interval.

Summary

The main contributions of the paper are twofold. The first is the theoretical contribution. We provide a general framework to determine a bank’s optimal LTV ratio after considering the collateral value under the Knightian uncertainty. The second contribution is the application in practice. When banks provide collateral loans for SMEs, they should not only take into account the macroeconomic measures, but also make more scientific research on the loan enterprises, including the determination of the liquidity of the pledge and the uncertainties faced by them.

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