Elasticity Analysis of Orthotropic Plate under Concentrated Force

Pu-rong JIA, Yong-yong SUO, Qiang WANG and Lei-lei ZENG
Department of Engineering Mechanics, Northwestern Polytechnical University, Xi’an 710129, PR China
*Corresponding author

Keywords: Orthotropic Plate, Concentrated Force, Complex Function, Stress Field.

Abstract. The complex variable function including a material parameter is analyzed fully. Typical boundary value problem is considered to the plane orthotropic materials. By constructing new stress function, the mechanic analysis for the plate carrying a concentrated force is carried out. The boundary problems of the basic partial equation are studied and the formulae for stress fields are derived in the rectangular and polar coordinates.

Introduction

The complex variable theory provides a very powerful tool for the solution of many boundary value problems in the elastic body. Such theory was originally found by Russian researchers for solving general boundary problems in isotropic materials [1~3]. Furthermore, the complex variable technique has also been expanded to use for anisotropic materials. Complex variable methods prove to be very useful for the solution of many full-space and half-space problems. Fiber-reinforced polymer matrix materials are the most typical composites, which are also as anisotropic materials at the macroscopic level [4, 5]. The orthotropic plate may have been the base of composites in common engineering use. Typical half-space examples include concentrated force and moment systems applied to the free surface. The feasible method to solve stress-field problems in anisotropic composites is to use complex analytic function theory, and the results have been reported [6]. But the general solutions may have some weakness. So the purpose of this paper is to focus attention on a new solution of the boundary-value problem for the orthotropic plate.

Concentrated Force on Straight Boundary

Consider now the half plane carrying a general concentrated force $P$ on a horizontal straight boundary surface of an infinitely large plate as shown in Figure 1. The direction of inclined force $P$ is defined by the angle $\varphi$ between $P$ and free straight boundary. The distribution of the load along the thickness of the plate is uniform. The thickness of the plate is taken as unity, so that $P$ is the load per unit thickness. The general inclined force $P$ can be resolved into two components, which are $P\cos\varphi$ horizontally and $P\sin\varphi$ vertically.

Figure 1. Scheme of the force acting on straight boundary.
The elasticity analysis of the plane stress problem is of great importance to the usual engineering application. The distribution of any stress depends on the forces acting on the complete closed boundary. It is supposed that the remote boundary is constrained (Figure 1), so that the stress is infinitely small at the place far away from the point of applied force. For a general isotropic material, the solution of stress distribution in Figure 1 can be found out from a book on the theory of elasticity. But for an anisotropic material, the solution of the typical problem is hard to find in books or articles. Therefore, the aim of this article is to give an example of showing the method to solve the boundary loading problem particularly for an orthotropic plate.

### Basic Equations

The plane stress state of composite sheets is common and very importance for the application. It is the key point to solve stress-field problems in orthotropic materials. Suppose the principal elastic directions of the plate coincide with the coordinate directions (x, y), and let the directions 1, 2 parallel to the axes x, y, respectively. Linear elastic strain-stress relations are known generally as Hooke’s law, and the linear constitutive equations for the orthotropic materials are given as follows:

\[
\varepsilon_x = \frac{\sigma_x}{E_1} - \frac{\nu_{12} \sigma_y}{E_1}, \quad \varepsilon_y = \frac{\sigma_y}{E_2} - \frac{\nu_{12} \sigma_x}{E_2}, \quad \gamma_{xy} = \frac{\tau_{xy}}{G_{12}}
\]

It is well known that the compatibility condition of strains must satisfy as follows:

\[
\frac{\partial^2 \varepsilon_x}{\partial y^2} + \frac{\partial^2 \varepsilon_y}{\partial x^2} = \frac{\partial^2 \gamma_{xy}}{\partial x \partial y}
\]

In the case of plane stress state, the equilibrium equations are as (body forces are absent):

\[
\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} = 0, \quad \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} = 0
\]

Usually, the method of solving the equations is by introducing a new function \( U \) of x and y, called the stress function. Through taking any real function \( U \), it is easily checked that the equilibrium equations are satisfied by putting the following expressions for the stress components:

\[
\sigma_x = \frac{\partial^2 U}{\partial y^2}, \quad \sigma_y = \frac{\partial^2 U}{\partial x^2}, \quad \tau_{xy} = -\frac{\partial^2 U}{\partial x \partial y}
\]

By means of using above relations, the governing equation of the compatibility condition can be expressed by the stress function \( U(x, y) \), which is

\[
\frac{\partial^4 U}{\partial y^4} + \left( \frac{E_1}{G_{12}} - 2\nu_{12} \right) \frac{\partial^4 U}{\partial x^2 \partial y^2} + \frac{E_1}{E_2} \frac{\partial^4 U}{\partial x^4} = 0
\]

For the isotropic case, this basic equation is reduced to

\[
\frac{\partial^4 U}{\partial y^4} + 2 \frac{\partial^4 U}{\partial x^2 \partial y^2} + \frac{\partial^4 U}{\partial x^4} = 0 \quad \text{i.e.,} \quad \nabla^2 \nabla^2 U = 0
\]

This is the biharmonic equation of conventional elasticity. Thus, the solution of a plane problem can be reduce to finding a solution of equation (5) that must also satisfy the boundary conditions.

### Complex Function and Stress

In order to solve the boundary loading problem about an orthotropic plate, the complex function shall be introduced for the convenience of investigation. The basic complex variable \( z = x + iy \) and its
conjugate \( \bar{z} = x - iy \) are normally appeared in books \((i = \sqrt{-1})\). Now, another complex variable \( w \) and its conjugate \( \bar{w} \) are selected and used, which are defined as:

\[
w = x + iby, \quad \bar{w} = x - iby
\]

in which \( b \) is a real constant. The derivative relation and some operations must be as follows:

\[
\frac{\partial w}{\partial x} = \frac{\partial \bar{w}}{\partial x} = 1, \quad \frac{\partial w}{\partial y} = ib, \quad \frac{\partial \bar{w}}{\partial y} = -ib, \quad w + \bar{w} = 2x, \quad w - \bar{w} = 2iby
\]

\[
w^2 + \bar{w}^2 = 2x^2 - b^2y^2, \quad w^2 - \bar{w}^2 = 4ibxy, \quad w\bar{w} = x^2 + b^2y^2
\]

The complex function \( \ln w \) shall be considered firstly to use for solving the boundary loading problem as shown in Figure 1. In terms of the boundary condition and experimental knowledge, the real stress function \( U \) can be determined by the form:

\[
U = i(Cx + Dy)(\ln w - \ln \bar{w})
\]

where \( C \) and \( D \) are arbitrary constants. The derivatives of \( U \) with respect to \( x \) or \( y \) are given by:

\[
\begin{align*}
\frac{\partial U}{\partial x} &= iC (\ln w - \ln \bar{w}) + i(Cx + Dy) \left( \frac{1}{w} - \frac{1}{\bar{w}} \right) \\
\frac{\partial U}{\partial y} &= iD (\ln w - \ln \bar{w}) - b(Cx + Dy) \left( \frac{1}{w} + \frac{1}{\bar{w}} \right)
\end{align*}
\]

\[
\begin{align*}
\frac{\partial^2 U}{\partial x^2} &= 2iC \left( \frac{1}{w} - \frac{1}{\bar{w}} \right) - i(Cx + Dy) \left( \frac{1}{w^2} - \frac{1}{\bar{w}^2} \right) \\
\frac{\partial^2 U}{\partial y^2} &= -2bD \left( \frac{1}{w} + \frac{1}{\bar{w}} \right) + ib^2 (Cx + Dy) \left( \frac{1}{w^2} - \frac{1}{\bar{w}^2} \right) \\
\frac{\partial^2 U}{\partial x \partial y} &= iD \left( \frac{1}{w} - \frac{1}{\bar{w}} \right) - bC \left( \frac{1}{w} + \frac{1}{\bar{w}} \right) + b(Cx + Dy) \left( \frac{1}{w^2} + \frac{1}{\bar{w}^2} \right)
\end{align*}
\]

Then on the basis of above relations, all three stress components can be expressed as:

\[
\begin{align*}
\sigma_x &= \frac{\partial^2 U}{\partial y^2} = 4b^2C \frac{x^2y}{(w\bar{w})^2} - 4bD \frac{x^3}{(w\bar{w})^2} \\
\sigma_y &= \frac{\partial^2 U}{\partial x^2} = 4b^2C \frac{y^3}{(w\bar{w})^2} - 4bD \frac{xy^2}{(w\bar{w})^2} \\
\tau_{xy} &= -\frac{\partial^2 U}{\partial x \partial y} = 4b^2C \frac{xy^2}{(w\bar{w})^2} - 4bD \frac{x^2y}{(w\bar{w})^2}
\end{align*}
\]

It is evident that the boundary conditions on the straight edge are also satisfied because \( \sigma_y \) and \( \tau_{xy} \) are zero along upper edge of the plate \((y = 0)\), which is free from external forces except at the point of the load application \((r = 0)\) as shown in Figure 1. Furthermore, according to above derivatives of the stress function, the governing equation (5) can be given by:
This equation is easily written in the form

\[ 8b^3 D \left( \frac{1}{w^3} + \frac{1}{w^3} \right) - 6ib^4 (Cx + Dy) \left( \frac{1}{w^4} - \frac{1}{w^4} \right) + \left( \frac{E_1}{G_{12}} - 2v_{12} \right) \left[ -4bD \left( \frac{1}{w^3} + \frac{1}{w^3} \right) - 4ib^2 C \left( \frac{1}{w^3} - \frac{1}{w^3} \right) + 6ib^2 (Cx + Dy) \left( \frac{1}{w^4} - \frac{1}{w^4} \right) \right] + \frac{E_1}{E_2} \left[ 8iC \left( \frac{1}{w^3} - \frac{1}{w^3} \right) - 6i(Cx + Dy) \left( \frac{1}{w^4} - \frac{1}{w^4} \right) \right] = 0 \]

The solution of the characteristic equation (12) is given as:

\[ b^2 = \frac{E_1}{2G_{12}} - v_{12} = \sqrt{\frac{E_1}{E_2}} \]

This is the need for the constant \( b \). It is suitable for \( b \) to take the positive real root. So that is:

\[ b = \sqrt{\frac{E_1}{2G_{12}} - v_{12}} = \sqrt{\frac{E_1}{E_2}} \]

### Solution of the Boundary Problem

The arbitrary constants \( C \) and \( D \) in equation (10) have been not yet determined. Therefore, some relative conditions must be considered. The key condition is that the sum of the forces distributed over any cross section of the plate should be in equilibrium with the external force \( P \). In a general way, the equilibrium relation of whole forces can be established by both horizontal direction and vertical direction respectively. In a simple and clear method, that is to take a regular cross section \( (y = 1) \).

Then the stress components \( \sigma_y \) and \( \tau_{xy} \) on the section are given by:

\[ \sigma_y = \frac{4b^3 C}{(x^2 + b^2)^2} - \frac{4bDx}{(x^2 + b^2)^2}, \quad \tau_{xy} = \frac{4b^3 Cx}{(x^2 + b^2)^2} - \frac{4bDx^2}{(x^2 + b^2)^2} \]

The equations of equilibrium for forces in the x and y directions are:
The complex variables can be written as:

\[ \tau_r, \tau_\theta, \sigma \]

are zero. The radial normal stress in the polar coordinate system can be obtained:

\[ \tau_r = ib \tau_\theta \]

Next, it is considered to take the relations between stresses in the two coordinate systems. It is common knowledge that the stresses can be expressed by the following relations:

\[ \sigma_r = \sigma_x \cos^2 \theta + \sigma_y \sin^2 \theta + 2 \tau_{xy} \sin \theta \cos \theta \]
\[ \sigma_\theta = \sigma_x \sin^2 \theta + \sigma_y \cos^2 \theta - 2 \tau_{xy} \sin \theta \cos \theta \]
\[ \tau_{r\theta} = (\sigma_x - \sigma_y) \sin \theta \cos \theta + \tau_{xy} (\cos^2 \theta - \sin^2 \theta) \]

Substituting the stresses of the expression (16) into the above (18), and to simplify the relations, then the stress components \( \sigma_r, \sigma_\theta, \tau_{r\theta} \) in the polar coordinate system can be obtained:

\[ \sigma_r = \frac{-2P \sin \varphi}{\pi} \frac{b^3 \sigma_{xy}^2}{(x^2 + b^2 y^2)^2} - \frac{2P \cos \varphi}{\pi} \frac{bx^3}{(x^2 + b^2 y^2)^2} \]
\[ \sigma_\theta = \frac{-2P \sin \varphi}{\pi} \frac{b^3 y^3}{(x^2 + b^2 y^2)^2} - \frac{2P \cos \varphi}{\pi} \frac{bxy^2}{(x^2 + b^2 y^2)^2} \]
\[ \tau_{r\theta} = \frac{-2P \sin \varphi}{\pi} \frac{b^3 xy^2}{(x^2 + b^2 y^2)^2} - \frac{2P \cos \varphi}{\pi} \frac{bx^2 y}{(x^2 + b^2 y^2)^2} \]

Acknowledgements

The authors acknowledge the financial support of the Natural Science Foundation of China (NSFC Grant No. 51475372).

References


