Parametrically Excited Stability of a Simply Supported Beam under Axial Periodic Excitation

Li FAN and Zu-guang YING

Department of Mechanics, School of Aeronautics and Astronautics, Zhejiang University, Hangzhou 310027, China

*Corresponding author

Keywords: Parametrically excited stability, Beam, Axial periodic excitation, General periodic force.

Abstract. The parametrically excited stability of beams with multiple mode vibration under general periodic axial excitation is studied and the periodic transverse supports are considered for the first time. The partial differential equation of motion of the beam with spaced supports under axial excitation is given and then converted into ordinary differential equations with periodic parameters by using the Galerkin method. The direct eigenvalue analysis method is applied to solve the parametrically excited stability of the beam described by the differential equations with periodic parameters based on the Fourier expansion and generalized eigenvalue analysis. A track beam with spaced supports under periodic axial excitation is considered. Numerical results on unstable regions are given to illustrate the parametrically excited stability of the beam and the influence of supports and excitation on the stability.

Introduction

The dynamic stability of periodically parametrically excited systems is an important research subject in engineering. For example, the parametrically excited vibration of cables in a cable-stayed bridge under support motion excitation can result in the cable instability and damage [1-3]. The parametrically excited vibration of beams under axial periodic excitation can result in the dynamic buckling [4-5]. The periodically parametrically excited systems can be expressed as the Hill equations or Mathieu equations. The stability of single Mathieu equation representing single-degree-of-freedom system has been quite studied by using the Floquet theory [6]. Several approximation methods for solving the stability of coupled Mathieu equations representing multi-degree-of-freedom system have been proposed. Based on the Floquet theory and harmonic balance method, the direct eigenvalue analysis method for the parametrically excited stability have been developed and applied to inclined stay cables with multi-degree-of-freedom to obtain unstable regions [7-9]. The parametrically excited stability of beams under axial periodic excitation has also been studied [4-5]. However, the stability analysis was based on single or several mode vibration and harmonic axial excitation. The parametrically excited stability of beams with multiple mode vibration under general periodic excitation needs to be studied further, which has a more practical significance in engineering.

The present paper mainly focuses on the parametrically excited stability of a beam with coupled multiple mode vibration under general periodic axial excitation, and the periodic supports for improving the beam stability are considered. First, the differential equation of motion of the beam with spaced supports under axial excitation is given. The Galerkin method is applied to convert the partial differential equation into ordinary differential equations. Second, the direct eigenvalue analysis method is applied to solve the parametrically excited stability of the beam with spaced supports under axial excitation described by the differential equations with periodic parameters. Finally, a track beam with spaced supports under periodic axial excitation is considered. Numerical results on unstable regions are given to illustrate the parametrically excited stability of the beam and the influence of supports and excitation on the stability.

310
Vibration Equation and Stability of Beam

For example, a railway track with lateral fasteners under periodic axial loading can be modeled as parametrically excited beam. Based on the Euler-Bernoulli beam theory, the differential equation of motion of the beam with spaced supports under axial excitation can be expressed as

$$\rho A \ddot{w} + c_b \dot{w} + \frac{EI}{L^4} \dddot{w} + F_x(t) \dddot{w} + \sum_{k=1}^{N_p} (c_{sk} \dot{w} + k_{sk} w) \delta(x - x_{sk}) = 0,$$

where $w$ is the beam displacement, $\rho$ is the mass density, $A$ is the cross-sectional area, $E$ is the elastic modulus, $I$ is the second moment of area, $c_b$ is the damping coefficient of beam, $F_x(t)$ is periodic axial excitation, $x$ is the axial coordinate, $k_{sk}$ is the stiffness of support $k$, $c_{sk}$ is the damping coefficient of support $k$, $x_{sk}$ is the coordinate of support $k$, $\delta(\cdot)$ is the Dirac delta function, $N_p$ is the support number. Eq. 1 can be rewritten in the dimensionless form

$$\dddot{v} + \frac{c_b}{\rho A} \dddot{v} + \frac{EI}{\rho AL^4} \dddot{v} + F_x(t) \dddot{v} + \frac{L}{\rho A} \sum_{k=1}^{N_p} (c_{sk} \dot{v} + k_{sk} v) \delta(z - z_{sk}) = 0,$$

where $v = w/lh$, $z = x/L$, $L$ is the beam length, $h$ is the cross-sectional characteristic size. The displacement $v$ can be expanded into

$$v(z,t) = \sum_{j=1}^{n} q_j(t) \phi_j(z),$$

where $q_j(t)$ is the time function, $\phi_j(z)$ is the shape function, $n$ is an integer. For the simply supported beam, $\phi_j(z) = \sin(j\pi z)$. According to the Galerkin method, substituting Eq. 3 into Eq. 2, multiplying the equation with $\sin(i\pi z)$ and integrating it with respect to $z$ yield differential equations for $q_j$. They are rewritten in the matrix form

$$M \dddot{Q} + C \dot{Q} + K(t)Q = 0,$$

where $Q = [q_1, q_2, \ldots, q_n]^T$ is the generalized displacement vector, $M$, $C$ and $K$ are respectively the generalized mass, stiffness and damping matrices which are determined by $\phi_j(z)$. $K(t)$ is the periodic function due to $F_x(t)$. Eq. 4 describes the parametrically excited system with coupled multiple mode vibration.

Based on the parametrically excited stability analysis method in the reference [8], the general periodic excitation $F_x(t)$ with period $T = 2\pi/\omega$ ($\omega$ is frequency) is expanded into Fourier series as

$$F_x(t) = \frac{a_0}{2} + \sum_{j=1}^{m_p} (a_j \cos j\omega t + b_j \sin j\omega t) = A_0 + \sum_{j=1}^{m_p} A_j \sin(j\omega t + \Theta_j),$$

where $m_p$ is an integer. The stiffness matrix can be correspondingly expressed as

$$K(t) = \frac{K_{cb}}{2} + \sum_{j=1}^{m_p} (K_{ij} \sin j\omega t + K_{ij} \cos j\omega t).$$

Eq. 4 can be converted into the state equation and the coefficient matrix $B(t)$ can be expressed as Fourier series by using Eq. 6. Based on the Floquet theory, the displacement vector $Q$ can be expressed as the product of periodic component and exponential component, and the periodic component can be expanded into Fourier series. Substituting these series into the state equation and balancing each harmonic term yield algebraic equations which result in the matrix eigenvalue problem. Based on the eigenvalues, the stability of generalized displacement $Q$ and then beam displacement $v$ can be determined directly.
Parametrically Excited Instability Analysis

Consider a typical period-supported track beam with parameter value $\rho=7.85\times10^3\text{kg/m}^3$, $A=7.745\times10^{-3}\text{m}^2$, $l=5.24\times10^{-6}\text{m}^3$, $E=206\text{GPa}$, $k_{sk}=19.6\times10^6\text{N/m}$, $c_{sk}=60\times10^3\text{N}\cdot\text{m/s}$, $L=20\text{m}$, $c_b=0.002$. The constant part of $F_X(t)$ is $A_0=0.5F_{cr}$ where $F_{cr}$ is the critical load for buckling, and $\theta=0$. The first six natural frequencies of the beam without spaced support are 0.5230, 2.0920, 4.7069, 8.3678, 13.0747, 18.8276 Hz. Figures 1a and 1b show the unstable regions on the plane of excitation frequency ($\omega$) and amplitude ($A_1$ or $A_2$) for the beam without spaced support. The unstable regions are around the twice natural frequencies. The minimal value and interval of the unstable regions increase with excitation frequency. Then the parametrically excited instability for low excitation frequency needs to be considered more than that for high excitation frequency.

Figures 2a and 2b show the unstable regions on the plane of excitation frequency ($\omega$) and amplitude ($A_1$ or $A_2$) for the beam with one spaced support. By comparing Figure 2 with Figure 1, it is obtained that the unstable regions around twice odd-order natural frequencies are reduced remarkably by the spaced support. Figures 3a and 3b show the unstable regions on the plane of excitation frequency and amplitude for the beam with two spaced supports. By comparing Figure 3 with Figure 1, it is obtained that the unstable regions around twice $(3i+M)$-order natural frequencies ($M=1,2$ and $i=0,1,2,\ldots$) are reduced remarkably by the two spaced supports. It can be inferred that the unstable regions for low excitation frequency are reduced and the parametrically excited stability is improved by increasing spaced supports.

Figures 4a and 4b show the unstable regions on the plane of excitation frequency and amplitude for the beam with one and two spaced supports under periodic excitation with $A_1$ and $A_2$ ($A_1=2A_2$), respectively. By comparing Figure 4a with Figure 2a and Figure 4b with Figure 3a, it is obtained that the unstable regions for periodic excitation with only $A_1$ are similar to those for periodic excitation with $A_1$ and $A_2$. Under equal $A_1$, some small unstable regions are enlarged by $A_2$. Thus, the main unstable regions and parametrically excited stability of the beam are determined by the basic frequency of the periodic axial excitation.

![Figure 1](image1.png)

(a) excitation with only $A_1$ 
(b) excitation with only $A_2$

Figure 1. Unstable regions of beam without spaced support under periodic excitation with $A_1$ or $A_2$. 

312
Summary
The parametrically excited stability of beams with coupled multiple mode vibration under general periodic axial excitation has been studied by using the direct eigenvalue analysis method. The periodic supports for improving the beam stability have been considered. The partial differential equation of motion of the beam with spaced supports under axial excitation has been given and...
converted into ordinary differential equations with periodic parameters by using the Galerkin method. The direct eigenvalue analysis method for the parametrically excited stability of the beam with spaced supports under axial excitation has been developed based on the Fourier expansion and generalized eigenvalue analysis. The track beam with spaced supports under periodic axial excitation has been studied as an example. Numerical results have illustrated that (a) the unstable regions of the beam without spaced support are around the twice natural frequencies, and the minimal value and interval of the unstable regions increase with excitation frequency; (b) the unstable regions for low excitation frequency are reduced and the parametrically excited stability is improved by increasing spaced supports; (c) the main unstable regions and parametrically excited stability of the beam are determined by the basic frequency of the periodic axial excitation.

Acknowledgement

The work was supported by the National Natural Science Foundation of China (No. 11572279).

References


