Optimal Control of a Stochastic Production-Inventory System under a Emission Rate Related Pollution Tax

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Keywords: Production-inventory, Pollution tax, Uncertainty, Optimal control.

Abstract. In this paper, we model a stochastic optimal control system of production-inventory under an emission rate related pollution tax. In our model, the inventory evolves according to a standard geometric Brownian motion process, and the evolution of emission in per unit output according to a determining dynamic process. Our objective is to apply optimal control theory to investigate the stochastic production-inventory system under uncertainties of inventory dynamics; and derives the optimal production level that maximizes the objective function value. The results are discussed with some illustrative examples for different demand rate functions.

Introduction

Production-inventory planning as one of the operations research and management science problems has received a considerable amount of attention. For example, Dobos (2003) presented optimal production-inventory strategies for a reverse logistics system, where the costs of this system consist of the linear holding costs for two stores and the convex non-decreasing manufacturing and remanufacturing costs, and there is no delay between the using and return processes. Alshamrani and El-Gohary (2011) extended the work of Dobos (2003) to apply the Pontryagin minimum principle to study the problem of optimal control of a two-item inventory system with different types of deterioration. The total cost, which includes the sum of the holding costs of inventory levels, the holding costs of one item due to the presence of the other and the production costs, is minimized. Liu et al. (2015) constructed an integrated production, inventory and preventive maintenance model for a multi-product production system to find the optimal number of production cycles per year and the optimal position of preventive maintenance that will maximize the expected profit per unit time.

Production processes lead to pollution. A key factor in environmental pollution control is the adoption of more efficient pollution abatement technologies by firms. Wirl(1991) gave the first insights in the integration of the environmental effects into inventory models. Using the optimal control theory, Li (2014) explored the introduction of banking carbon allowances. The authors proved that allowance banking causes higher inventory levels and a smoother behavior on production rate. Based on a stochastic model with random demand and environmental impacts over a finite-horizon, Chen and Monahan (2010) determined the optimal inventory policies, the authors proved that when organizations are working under a mandatory scheme, they tend to increase their inventory levels. Hammami et al. (2015) developed a deterministic optimization model that incorporates carbon emissions in a multi-echelon production-inventory model with lead time constraints. They demonstrated that individual caps can achieve significantly lower emissions but can paradoxically lead to increasing the per unit emissions and show how a share of emissions can improve per unit emissions without deteriorating total emissions. In 2015, García-Alvarado et al. extended their previous work to a finite-horizon.

To extend the studies of García-Alvarado et al. (2015), in this paper, a market-based environmental policy instruments is introduced into the decision-making process of production-inventory. We dealt with an infinite-horizon decision-making problem subject to a pollution tax scheme, but not an emission-cap or a trade scheme, where the tax is levied on the marginal
contribution to the accumulation pollution. Different from Li (2014) and García-Alvarado et al. (2015), in this paper, the amount of pollutants on per unit output is not a constant, we assume it obey a dynamic evolving process, in particular, it can be reduced through pollution abatement investment performed by the firms who are incentive by the pollution tax scheme. Different from Li (2014), in their studies, the determined inventory dynamic was considered, however, in this paper, the uncertainty of inventory dynamic evolution is incorporated into our model, we assume it obeys a geometric Brownian motion.

The Basic Modal

Let us consider a firm that produces a single item, selling some units and storing the other units. We assume that the production deteriorates while in stock and the demand rate varies with time. Let \( r \) be the risk-free discount rate; \( h \) be the linear inventory holding costs; \( R(t) \) be the production level at time \( t \); \( C(R(t)) \) be the production cost at time \( t \), non-decreasing, strictly convex function, non-negative; \( D(t) \) be the known determinative demand rate at time \( t \); \( I(t) \) be the inventory status in point \( t \), we assuming that the production deteriorates while in stock and the demand rate varies with time, furthermore, the dynamics of the inventory is affected by stochastic shocks giving rise to a geometric Brownian motion process:

\[
dI(t) = [R(t) - D(t) - \theta I(t)]dt + \sigma I(t)d\zeta(t)
\]

where the initial condition is \( I(0) = I_0 \) and the parameter \( \theta \) is the natural deterioration rate; \( \sigma \) is the diffusion coefficient. \( \zeta(t) \) is the standard Wiener process, which can be expressed as \( w(t)dt \), where \( w(t) \) is a white noise process. All functions are assumed to be non-negative, continuous and differentiable.

To create an incentive for the firms to invest in pollution reduction projects so as to improve its productive technology to be more environment-friendly, following Dragone et al.(2010), We assume that the firm is imposed an instantaneous pollution tax equal to \( \tau b(t) \), i.e., what is being taxed is indeed the rate \( b(t) \) at which a unit of final product contributes to the increase in the stock of pollution, where \( \tau \) denotes the pollution tax rate in \( b(t) \) which accumulates according to the dynamics

\[
db(t) = (-k(t) + \eta b(t))dt
\]

where the initial condition is \( b(0) = b_0 \), \( b(t) \) decreasing in \( k(t) \geq 0 \), which is the instantaneous investment in pollution abating technology carried out by the firm, \( \eta > 0 \) is a constant, a plausible economic interpretation of \( \eta > 0 \) is which can be seen as the environmental obsolescence rate of technologies, measuring the growth rate of the external damage produced by the use of technologies that become more polluting as time goes by. Assuming the pollution abatement investment cost is given by

\[
\Gamma(t) = \gamma k^2(t)
\]

where \( \gamma \geq 1 \) represents the efficient measure of emissions abatement investment. Since production cost function \( C(R(t)) \) is non-decreasing, strictly convex function, non-negative, thus, it can be written as the following quadratic form:

\[
C(R(t)) = cR^2(t)
\]

where \( c \) is a constant.

After introducing the uncertainties of inventory dynamics, the stochastic production-inventory model in infinite planning horizon can be written as:
where $E$ is an expected operator, $P(t)$ is the product price at time $t$ and we assume it a constant. The model (5) is represented as a stochastic optimal control problem, in which the control variables of the model are the schedule production level $R(t)$ and the investment level $k(t)$; the state variables are the inventory level $I(t)$ and the emission levels in per unit product $b(t)$.

### The Solution of the Stochastic Production Inventory Model

In order to solve the model (5), we apply the Hamilton–Jacobi–Bellman equation (HJB). Let $V(I,b)$ be the expected value of the objective function at time $t$ in infinite planning horizon, then the HJB equation for the maximization problem faced at time $t$ is

\[
\begin{align*}
    rV(I,b) - \frac{1}{2} I^2 \sigma^2 V_{\mu,\mu}(I,b) = \max_{R(t),k(t)} & \left\{ DP[hI + cR^2 + \tau b + \gamma k^2] + V_r(I,b)R - D - \theta I + V_h(I,b)\left[-k + \eta b\right] \right\}.
\end{align*}
\]

Performing the indicated maximization in (6) yields:

\[
\begin{align*}
    R^* = \frac{V_r(I,b)}{2c}, \quad k^* = \frac{V_h(I,b)}{2\gamma}.
\end{align*}
\]

Substituting Eq. (7) into Eq. (6), we obtain

\[
\begin{align*}
    rV(I,b) = \frac{1}{2} I^2 \sigma^2 V_{\mu,\mu}(I,b) + \frac{1}{4c} V_{\mu,\mu}^2(I,b) + \frac{1}{4\gamma} V_{h,h}(I,b) - (D + \theta I)V_r(I,b) + \eta b V_h(I,b) - DP - hI - \tau b.
\end{align*}
\]

In order to get the solution of Eq. (8), we assume $V(I,b)$ has the form as following:

\[
\begin{align*}
    V(I,b) = Q_I I^2 + Q_Z b^2 + Z_I I + Z_Z b + M
\end{align*}
\]

where $Q_I, Q_Z, Z_I, Z_Z$ and $M$ are undetermined constants. Solving Eq. (9), one obtains:

\[
\begin{align*}
    V_r(I,b) &= 2Q_I I + Z_I, \quad V_h(I,b) = 2Q_Z b + Z_Z, \quad V_{\mu,\mu}(I,b) = 2Q_I, \quad V_{h,h}(I,b) = 2Q_Z, \quad V_{b,b}(I,b) = 2Q_Z b + Z_Z.
\end{align*}
\]

Substituting Eq. (9)-(13) into (8), we obtain
\[ Q_i = c(r - \sigma^2 + 2\theta) \quad Q_j = \gamma(r - 2\eta) \]  

\[ Z_1 = 2cQ_iD + ch \quad Z_2 = \frac{\gamma\tau}{\gamma\eta + Q_2 - r\gamma} \]  

\[ M = \frac{1}{r} \left( \frac{Z_1^3}{4c} + \frac{(Z_2)^3}{4\gamma} - DZ_1 + DP \right) \]  

Substituting Eq. (10) and (12) into (7), the optimal control problem Eq. (5) is then

\[ R^* = \frac{2Q_1I + Z_1}{2c} \quad k^* = -\frac{2Q_2b + Z_2}{2\gamma} \]  

Substituting the optimal solution of pollution abatement investment \( k^* \) from Eq. (18) to (2), the determined the pollution emissions dynamic is obtained:

\[ b^*(t) = (b_0 + \frac{Z_2}{2(\gamma\eta + Q_2)})e^{\frac{(\eta - \theta)b_0}{\gamma}} - \frac{Z_2}{2(\gamma\eta + Q_2)} \]  

Similarly, substituting the optimal solution of production decisions \( R^* \) from Eq. (20) to (1), we get the expected inventory dynamic:

\[ I^*(t) = (I_0 + \frac{Z_1}{2(Q_1 - c\theta)})e^{\frac{(\theta - \theta)^2}{c}} - e^{\frac{(\theta - \theta)^2}{c}} \int De^{\frac{(-\theta - \theta)t}{c}} dt - \frac{Z_1}{2(Q_1 - c\theta)} \]  

### Numerical Examples

We have obtained the expression of solution of the model. In this section, we will investigate the numerical solutions of the model to determine an optimal control strategy under the following three cases. We assume there are three kinds of demand rates such as linear, quadratic and exponential increasing demand rate respectively. The parameters used in the numerical examples are presented in Table 1 and we use the version 7.0 of the Wolfram Mathematica Matlab to obtain the numerical solutions.

<table>
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<th>Parameter</th>
<th>0.1</th>
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<th>0.022</th>
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Suppose that there are three kinds of demand rates such as linear, quadratic and exponential increasing demand rate, i.e., \( D(t) = 20 + t \), \( D(t) = 20 + t^2 \) and \( D(t) = 20 + e^t \) respectively. Given this demand rate, the numerical solutions of the model (5) are displayed in figures 1-5.
Figure 1-5 show that under three kinds of demand rate such as linear, quadratic and exponential increasing demand rate, i.e., $D(t) = 20+t$, $D(t) = 20+t + t^2$, and $D(t) = 20+e^t$ respectively, the optimal expected inventory level $I(t)$, production level $R(t)$, and the value of the objective function $V(t)$ continue to rise with time and the rate of increase is significantly positively related to the demand function. Note that here demand and deterioration does not decrease the inventory level displayed in figure 1. We also see from above numerical solutions, the changes of demand function have no effect on pollution abatement investment and emissions level in per unit product and because of the pollution abatement investment maintain at a certain level, the pollution emissions level in per unit product come down.

**Conclusions**

In this paper, we extend García-Alvarado et al. (2015)’s model to an even more general model in which the uncertainty of inventory evolution, deteriorating items, emission tax and pollution abatement investment are taken into account. In our paper, the significant feature is that the emissions tax is levied on the firm’s emissions level in per unit product other than the total amount of the environmental externality. The aim of this paper is to apply the HJB equation to determine an optimal control strategy for a stochastic production-inventory system with deteriorating items, emission tax and pollution abatement investment.

Further, the optimal solution of the control system is presented numerically for different cases of demand rate. We find that: (i) the increasing of demand and deterioration does not decrease the inventory level; (ii) changes of demand function have no effect on pollution abatement investment and emissions level in per unit product. (iii) pollution abatement investment maintains at a certain level, and which lead to the pollution emissions level in per unit product come down.

The present paper has discussed the optimal control of a stochastic production-inventory system with deteriorating items, emission tax and pollution abatement investment. However, the uncertainty in emission-abatement system is not taken into account. Typically, there may be uncertainty in the evolution of the pollution stock and pollution abatement cost. A further research direction would be needed to examine these situations.

**Acknowledgement**

The authors would like to acknowledge the financial support from the Hunan Natural Science Foundation, China (Project no. 2018JJ2335) for this paper.
References


