Optimized Least Square DOA Estimation Algorithm based on Phase Interferometer Array

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Abstract. Aiming at the complex process of phase interferometer ambiguity resolution and the strict requirement of baseline comparison, an optimized least square ambiguity resolution algorithm for multi-baseline phase interferometer is proposed. This algorithm is a backward-forward derivation method. First, phase matrix is generated by off-line calculation. Then verify the uniqueness condition of correctly ambiguity resolution, and the angle of arrival (AOA) is solved by criterion function. The algorithm uses the mean square operation instead of the time-consuming cosine function calculation. The algorithm can be applied to the complex array of interferometers and makes the test of the uniqueness condition simpler. Simulation experiments are carried out under the condition of one-dimensional linear array. The result show that the algorithm is effective and has high anti-noise performance.

Introduction

Phase interferometer is widely used in the measurement and control of direction of arrival (DOA) in passive location, and it is of great significance in DOA Location and tracking [1]. The phase interferometer has the advantages of high direction-finding accuracy, passive direction finding, simple internal structure and observation frequency bandwidth [2]. With the improvement of single satellite information processing ability, phase interferometer is widely used in passive location. The phase detection range of phase interferometer is (-π, π). When the length between two baselines exceeds half wavelength (λ/2), phase ambiguity will occur. Therefore, for single baseline phase interferometer, there is a contradiction between direction finding accuracy and maximum non ambiguity angle [3,4]. In reference [5], the long-short baseline method limits the length of the short baseline. In order to meet the requirement of wide-band direction finding of phase interferometer and ensure the realization of system physics, a method based on virtual baseline is proposed in reference [6,7]. The construction of virtual baseline is obtained by subtracting adjacent entity noise-added baseline, so the virtual baseline will increase the disturbance of noise. Moreover, the virtual baseline will limit the placement of two long baselines adjacent to the short baseline. The reference [8] puts forward multi-pare unwrap ambiguity algorithm. The baseline is required to be mutual prime. The phase ambiguity can be obtained by two-dimensional integer search, but each group is required to contain the correct ambiguity number. The reference [9] puts forward stagger-baseline algorithm and require the baselines satisfied stagger qualification. The ambiguity number of each baseline can be obtained by multi-dimensional integer search. In reference [10], the cosine function is used to eliminate the resolution of the ambiguity number. The operation of the algorithm is mainly concentrated in the calculation of multiple cosine functions. In reference [11], a high-precision AOA algorithm is introduced, and the maximum allowable phase error is analyzed. In the case of multi-baseline, the phase value of the longest baseline needs to be obtained through the above processes more times. In reference [12], direction finding with uniform circular array (UCA), searching for ambiguity number with look-up table, and then calculating DOA with ambiguity number.

In this paper, an optimized least square DOA estimation algorithm is proposed. The algorithm is based on the fact that the phase difference corresponding to each incoming wave angle is constant in the case of noiseless. A matrix can be built offline to store the phase difference at each angle, and a
criterion function can be built to solve the DOA. This algorithm has three advantages compared with above methods. First, it avoids the process of solving the ambiguity number, does not need multi-dimensional integer search, and has a small amount of calculation. Second, the algorithm generates the phase matrix, which makes the test of uniqueness condition simpler. Third, the algorithm is not only applicable to linear array, and it is suitable for plane array, such as L array and UCA.

DOA Estimation Algorithm Based on Least Square Criterion

One Dimensional Linear Phase Interferometer Array

Suppose that there is a phase interferometer array with N+1 elements distributed on a one-dimensional straight line, the baseline $A_0$ is defined as the reference baseline of phase 0, and the distance between $A_1 \sim A_N$ is $D_1, D_2, \cdots, D_N$, as shown in Fig. 1. The radiation source is in the far-field position relative to the interferometer, the wavelength is $\lambda$, the angle of the incoming wave is $\theta$, and the phase difference between the baseline $A_1 \sim A_N$ and $A_0$ is:

$$\phi_n = \frac{2\pi D_n}{\lambda} \sin(\theta), n = 1, 2, \cdots, N$$

(1)

The actually received phase difference $\varphi_n$ differs from $\phi_n$ by a number of $2\pi$:

$$\varphi_n + 2\pi M_n = \phi_n, \varphi_n \in [-\pi, \pi)$$

(2)

where $M_n$ is the phase ambiguity number between $A_n$ and $A_0$.

![Figure 1. One-dimensional N+1 baseline phase interferometer.](image)

In the case of noise, the observed phase $\varphi_n$ is the modulus of $2\pi$ after the phase noise $v_n$ is superimposed on $\varphi_n$:

$$\varphi_n = (\phi_n + v_n) \mod(2\pi)$$

(3)

Obviously, in the noiseless case, under a certain angle $\theta_0$, the value of $\varphi_1, \varphi_2, \cdots, \varphi_N$ are determined. According to this characteristic, we can store all the phase values in the interval $[-\theta, \theta]$ to generate a phase matrix. Then we can use the optimized least square criterion to solve the DOA.

Design of Phase Matrix

The construction of phase matrix is a very important step in DOA estimation. The matrix needs to enumerate all possible incident angles. Assuming the incident range of the incident angle is $[-\theta, \theta]$, the distance between each baseline and the reference baseline $A_0$ is $D_1, D_2, \cdots, D_N$. The frequency of the incoming wave is $f$. If the correct estimation of DOA is defined as the $\pm 1^\circ$ deviation of the
incoming wave direction, then \([-\theta, \theta]\) is evenly divided into M angles in the step of 0.1°, the matrix \(A_{M\times N}\) is generated by off-line calculation:

\[
A_{M\times N} = \begin{bmatrix}
\psi_1 \\
\psi_2 \\
\vdots \\
\psi_M
\end{bmatrix} = \begin{bmatrix}
\varphi_1(\theta) & \varphi_2(\theta) & \cdots & \varphi_N(\theta) \\
\varphi_1(\theta_2) & \varphi_2(\theta_2) & \cdots & \varphi_N(\theta_2) \\
\vdots & \vdots & \ddots & \vdots \\
\varphi_1(\theta_M) & \varphi_2(\theta_M) & \cdots & \varphi_N(\theta_M)
\end{bmatrix}
\]

(4)

where \(\psi_\theta\) is the row vector:

\[
\psi_\theta = [\varphi_1(\theta) \ \varphi_2(\theta) \ \cdots \ \varphi_N(\theta)] = \left(2\pi \times \frac{\sin(\theta)}{\lambda} \times [D_1 \ D_2 \ \cdots \ D_N]\right) \text{mod}(2\pi)
\]

(5)

The above is the generation process of the phase matrix \(A_{M\times N}\). If the step is smaller than the current step but does not increase the accuracy of the resolution, the current step is the appropriate step size. In high-precision direction finding, it is necessary to establish a small step length phase matrix, which will increase the length of the matrix. However, the phase matrix \(A_{M\times N}\) can be used for DOA estimation repeatedly and does not need to be regenerated every time.

**Optimized Least Squares Criterion**

Assuming the incident angle is \(\theta_0\), the phase of the interferometer with noise is \(\psi_0 = [\varphi_1, \varphi_2, \cdots, \varphi_N]\). Since the phase matrix \(A_{M\times N}\) between \([-\theta, \theta]\] has been generated, \(\psi_0\) must have the highest correlation with a row vector \(\psi_\theta\) between \(\psi_1, \psi_2, \cdots, \psi_M\). The \(\hat{\theta}\) corresponding to \(\psi_\theta\) can be calculated according to the least square criterion:

\[
\hat{\theta} = \arg \min_{\theta \in [-\theta, \theta]} |\psi_0 - \psi_\theta|^2 = \arg \min_{\theta \in [-\theta, \theta]} \left(\sum_{i=1}^{N} |\varphi_i - \varphi_i(\theta)|^2\right)
\]

(6)

This is the result of the least square criterion when \(\varphi_i\) and \(\varphi_i(\theta)\) are in the same period. Because the observation phase difference takes \(2\pi\) as the period, the phase difference near the junction of \(-\pi\) and \(\pi\) may enter the previous or next period after the noise is superposed. Assume that the noiseless phase difference is \(\phi = \varphi + 2\pi M\), and the phase difference after adding noise is \(\phi' = \varphi + 2\pi M + \nu\), the noiseless observation phase is \(\varphi\):

\[
\varphi = (\varphi + 2\pi M) \text{mod}(2\pi) = \varphi, \varphi \in [-\pi, \pi)
\]

(7)

The noise-added observation phase is:

\[
\varphi' = (\varphi + 2\pi M + \nu) \text{mod}(2\pi) = \begin{cases}
\varphi + \nu, & -\pi \leq \varphi + \nu < \pi \\
\varphi + \nu + 2\pi, & -2\pi \leq \varphi + \nu < -\pi \\
\varphi + \nu - 2\pi, & \pi \leq \varphi + \nu < 2\pi
\end{cases}
\]

(8)

Three situations can happen. In fact, the noise observation phase superimposes the noise \(\nu\) only on the noiseless observation phase. Therefore, it is necessary to modify the least square criterion. Set \(\min(|\varphi - \varphi'|, 2\pi - |\varphi - \varphi'|)\) as the minimum value of the two phase, then the difference between the modified noise phase and the noiseless phase is:
\[ |\varphi' - \varphi| = \min \left( \min \left( |\varphi' - \varphi|, 2\pi - |\varphi' - \varphi| \right), \min \left( |\varphi' - \varphi|, -2\pi - |\varphi' - \varphi| \right), \min \left( |\varphi' - \varphi|, \pi - |\varphi' - \varphi| \right) \right) \]

It can be seen that the optimized least square criterion reflects the superimposed error \( v \), and the DOA is estimated as follows:

\[
\hat{\theta} = \arg \min_{\theta \in [\alpha, \beta]} \left( \sum_{i=1}^{N} \min \left( |\varphi_i(\theta)|, 2\pi - |\varphi_i(\theta)| \right)^2 \right) \]

(10)

Analysis of the Uniqueness Conditions for Correctly Ambiguity Resolution

If any two angles \( \alpha \) and \( \beta \) within \([-\theta, \theta]\), \( \alpha \neq \beta \), the result is \( \hat{\alpha} = \hat{\beta} \) according to the optimized least squares criterion. If \( \hat{\alpha} = \hat{\beta} \) is solved regardless of the noise \( v \), the problem of the uniqueness conditions arises and the phase interferometer array needs to be re-adjusted. If \( \hat{\alpha} = \hat{\beta} \), then:

\[
\arg \min_{\alpha \in [a, b]} \left( \sum_{i=1}^{N} \min \left( |\varphi_i(\alpha)|, 2\pi - |\varphi_i(\alpha)| \right)^2 \right) = \arg \min_{\beta \in [a, b]} \left( \sum_{i=1}^{N} \min \left( |\varphi_i(\beta)|, 2\pi - |\varphi_i(\beta)| \right)^2 \right) \]

(11)

If it is not related to noise \( v \), after simplification:

\[
\varphi_i(\alpha) = \varphi_i(\beta), i = 1, 2, \cdots, N \]

(12)

The meaning is the row vector \( \psi_{a} = \psi_{b} \) in the phase matrix. So if the generated phase matrix \( A_{M \times N} \) has the same two row vectors, the phase interferometer array does not conform to the uniqueness condition.

Before the DOA estimation, it is necessary to verify the uniqueness condition of the ambiguity resolution. However, this algorithm is much simpler than other methods in verifying the uniqueness condition. After generating the phase matrix \( A_{M \times N} \), it is required that no two row vectors are the same, which is suitable for linear and plane phase interferometer arrays.

Algorithm Performance Analysis

The core of the algorithm is to deal with the phase matrix. If two different arrays have the same dimension of the phase matrix, the two arrays have the same amount of computation. When the frequency of the incoming wave and the position of the baseline are known, the algorithm needs to perform the following two steps for a phase interferometer array:

**Step 1.** The generation process of phase matrix. The phase matrix \( A_{M \times N} \) can be generated according to Eq. 4 and Eq. 5, then the uniqueness condition needs to be verified. If it is satisfied, the phase matrix is stored. The first step requires approximately \( M \times N \sin \) function calculations and \( M \times N \mod \) function calculations, and in the case where the incoming wave frequency and the baseline position are unchanged, the step 1 only needs to be performed once.

**Step 2.** The optimized least squares ambiguity resolution process. The incoming wave incident from the unknown direction, DOA can be solved according to Eq. 10. It is necessary to perform \( M \times N \) comparison calculations, \( M \times N \) square calculations, and \( 3 \times M \times N \) addition calculations. Finally, the DOA can be obtained by one-dimensional minimum search.

It can be seen that after generating the phase matrix \( A_{M \times N} \), the algorithm only needs to repeat the step 2 to calculate the DOA and does not need to generate the phase matrix every time. Therefore, the
calculation amount of this algorithm is small and independent of the position of the baseline. After meeting the uniqueness condition, the position of the baseline is different, the accuracy is different, but the algorithm calculation amount will not increase for the same dimensional phase matrix, which is also an advantage of this algorithm. It can optimize the accuracy by using the position of baseline without increasing the calculation amount.

**Simulation Analysis**

In order to verify the effectiveness of the algorithm in one-dimensional linear array and the accuracy of the ambiguity resolution under the condition of noise, the simulation conditions as shown in Table 1 are used for simulation analysis:

<table>
<thead>
<tr>
<th>Parameter Name</th>
<th>Parameter Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Number of baselines</td>
<td>5</td>
</tr>
<tr>
<td>Signal frequency</td>
<td>10[GHz]</td>
</tr>
<tr>
<td>Baseline length</td>
<td>[0 11λ/2 23λ/2 42λ/2 81λ/2]</td>
</tr>
<tr>
<td>Unambiguous direction-finding range</td>
<td>−70° ~ 70°</td>
</tr>
<tr>
<td>Phase matrix step size</td>
<td>0.1°</td>
</tr>
<tr>
<td>Correctly DOA estimated deviation</td>
<td>±1°</td>
</tr>
</tbody>
</table>

After establishing the phase matrix \( A_{M \times N} \), it is easy to verify that the simulation condition satisfies the uniqueness condition of correctly ambiguity resolution under the condition of −70° ~ 70°. Assuming that the wave angle is −60°, −25°, −5°, 15°, 35°, 55°, verify the correctness of the ambiguity resolution under uniform noise and Gaussian noise.

Assuming that the noise \( v_n \) is independent of each other and uniformly distributed in the range of \([−q, q]\), \( q ≥ 0 \), perform 10000 simulations at each noise point \( q \) under the conditions shown in Table 1 to verify the algorithm performance as \( q \) changes, as shown in Fig. 2. Under the same experimental conditions, if the noise \( v_n \) is an independent Gaussian noise, the mean value is 0, and the variance is \( \sigma_v^2 \), and perform 10000 simulations at each noise point under the condition of Gaussian noise, as shown in Fig. 3. The simulation results under the two kinds of noise conditions show that the optimized least square algorithm has good anti-noise performance and high accuracy of ambiguity resolution.

![Figure 2. Linear array with uniform noise.](image1)

![Figure 3. Linear array with Gaussian noise.](image2)

**Summary**

This paper proposes an optimized least square DOA estimation algorithm, which avoids the complicated process of solving ambiguity number in the calculation. Criterion function takes full
account of the noise on each baseline and adopts the overall optimization to obtain the DOA. Under the same simulation conditions, the phase matrix can be used for direction finding repeatedly, which saves the computation. Under the MATLAB simulation experiment, the algorithm can correctly estimate the DOA of both linear and plane array phase interferometer, and has high anti-noise performance. This algorithm has important reference significance for the realization of practical engineering.

References