Computer Simulation of Systems of Axisymmetric Current-Carrying Superconducting Multi-Connected Bodies by the Method of Integral Equations

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ABSTRACT

The system of integral equations for surface currents of superconductors is obtained on the basis of the condition of magnetic flux conservation in the superconductor circuit for the axisymmetric system of superconducting bodies. The equations of the system have weakly singular kernels. A numerical method for solving model equations is developed using the method of singularity location. The study of the numerical model showed its correctness and convergence. However, there is a poor conditionality of the numerical model. On the example of the three-body problem, it is shown that with the number of sampling points about 100 on the contour of each body the calculation accuracy is 0.1%. It is noted that in comparison with the finite element method the number of degrees of freedom decreases by an order of magnitude.

Keywords: current-carrying superconducting systems, integral equations, computer simulation.

INTRODUCTION

Modeling of superconducting current-carrying structural elements of superconducting energy devices [1,2] can be carried out by means of integral equations for surface current densities [3-7].
The integral model can be theoretically constructed for an arbitrary system of superconducting bodies in the Meisner state on the basis of boundary conditions for the magnetic field at the body surface [8]:
\[
\vec{n} \times \vec{H}_t = \vec{i}, \quad H_n = 0,
\]
where \(\vec{i}\) – surface current density, \(\vec{n}\) – unit vector of the external normal to the surface of the superconductor. The expression of the magnetic field here through the surface current density using the known integral relations [9] leads to a system of integral equations for the surface current densities. However, the kernel of the obtained integral equations is the double layer potential, which gives a discontinuous solution on the integration surface. Therefore, the numerical solution of such equations encounters significant difficulties, practically excluding the possibility of their real use.

FORMULATION OF THE GENERAL MATHEMATICAL MODEL

The situation is significantly simplified in the case where it is possible to put boundary conditions directly on the vector potential \(\vec{A}\), since the core of the obtained integral equations will be the potential of a simple layer, and its solution will be a continuous function on the integration surface. This, in particular, is the case for a system of bodies obtained by rotating the system of their cross sections around one common axis \(O_z\), that is, the axisymmetric system of bodies. From the symmetry of the problem, in this case, it follows that in the cylindrical coordinate system, only the components \(H_\rho\) and \(H_z\) of the magnetic field and, accordingly, only the component \(A_\phi\) of the vector potential will be different from zero. The boundary condition for it on the surface of the \(k\)-body follows directly from the condition of preserving the magnetic flux in the superconductor circuit, that is, equal to zero of the magnetic field inside the superconductor [10]:
\[
A_\phi^{(k)} = \frac{\Phi_k}{2\pi\rho_k}.
\]
Here \(\Phi_k\) is flux through \(k\)-body contour, \(\rho_k\) is the radial cylindrical coordinate on the surface of this body.

In turn, the full vector potential can be represented as a sum of vector potentials of surface currents of all bodies [10]
\[ A_{\phi}^{(\text{int})} = \frac{\mu_0}{2\pi} \oint_{C} J_{\phi}(\rho', z') \sqrt{\frac{\rho}{\rho'}} f(m) dl', \quad (2) \]

and the vector potential of the external magnetic field, which, for example, for a uniform magnetic field parallel to the axis \( O_{\bar{z}} \), has the form

\[ A_{\phi}^{(\text{ext})} = \frac{\rho B_0}{2}, \quad (3) \]

where \( \mu_0 \) is permeability of vacuum, \( \bar{J} \) is surface current density, \( B_0 \) is induction of external homogeneous magnetic field,

\[ f(m) = \frac{1}{\sqrt{m}} \left[ (2-m)F(m) - 2E(m) \right], \quad (4) \]

Here \( F \) and \( E \) are full elliptic integrals of the first and second kind with the parameter

\[ m = \frac{4\rho\rho'}{(\rho + \rho')^2 + (z - z')^2}, \quad (5) \]

and \( C \) is the contour of the cross section of superconductor.

As a result, the system of integral equations for surface currents of superconductors converted to symmetric kernel will have the form:

\[ \sum_{n=1}^{K} \oint_{C_n} Q(l, l') J_n(l') dl' = \frac{\Phi_k}{\mu_0} - \frac{\pi \rho_k^2 B_0}{\mu_0}, \quad k = 1..K. \quad (6) \]

Here \( C_n \) is the contour of the cross section of the \( n \)-body, \( K \) is the number of connected bodies, and the kernel \( Q \) is determined by the formula

\[ Q(l, l') = \sqrt{\rho \rho'} f(m). \quad (7) \]

The system (6) is a system of Fredholm integral equations of the first kind. However, the kernel (7) has a logarithmic singularity when \( l = l' \), therefore, the usual methods of discretization to the system (6) are not applicable, and the development of a special algorithm for discretization of the system of integral equations is required.
SAMPLING AND STUDY OF THE MATHEMATICAL MODEL

On each contour $C_n$ we choose a system of $N_n$ equidistant points that divide the contour $C_n$ into $N_n$ segments. We denote the value of the surface current density in the middle of these segments as $J_i^{(n)}$.

To preserve the symmetry of the kernel when sampling we are to compute the kernel $Q(l,l')$ in the same points over two variables $l$ and $l'$. However, the kernel is increasing to infinity at $l \rightarrow l'$, but it is integrable in the neighborhood of this point. Therefore, we use the singularity extraction method [11] to approximate the integral in (6). As a result, we obtain a system of linear algebraic equations

$$
\sum_{n=1}^{K} \Delta l_n \sum_{i=0}^{N_n-1} Q_{j,l}^{(n,k)} J_i^{(k)} = \frac{\Phi_k}{\mu_0} - \frac{\pi (\rho_j^{(k)})^2 B_0}{\mu_0}.
$$

(8)

Here the matrix elements of the interaction of the currents are determined by the expressions

$$
Q_{i,j}^{(n,k)} = \begin{cases} 
\rho_j^{(k)} \left( \ln \frac{16 \rho_j^{(k)}}{\Delta l_k} - 1 \right), & i = j \land n = k \\
\sqrt{\rho_i^{(n)} \rho_j^{(k)}} f \left( \frac{4 \rho_i^{(n)} \rho_j^{(k)}}{(\rho_i^{(n)} + \rho_j^{(k)})^2 + (z_i^{(n)} - z_j^{(k)})^2} \right), & i \neq j \lor n \neq k 
\end{cases}
$$

(9)

The solution of the model (8.9) we perform on the example of a ring with a circular cross section with radius $a$, ring radius is $b$. The result of calculating the density distribution of the superconducting current along the perimeter of the ring section at the parameters $b = 20$ mm, $a = 10$ mm, $\Phi = -1 \cdot 10^{-7}$ Wb, $B_0 = 1.5 \cdot 10^{-4}$ T and $N = 100$ is shown in Figure 1 in polar coordinates.

The magnetic field induction was calculated using the formulas

$$
B_\rho = -\frac{\partial A_\varphi}{\partial z}, \quad B_z = \frac{\partial A_\varphi}{\partial \rho} + \frac{A_\varphi}{\rho},
$$

(10)

where the vector-potential component $A_\varphi$ is expressed by formulas (2) and (3). The result of the induction calculation is also shown in Figure 1.
Figure 1. The estimated distribution of the surface current density on the surface of the ring and the distribution of the magnetic field for a circular ring shape.

On the base of the results of the surface current calculation we can found macro-characteristics of the system – the coefficient of inductance $L$ and the force $F$ acting on given superconducting body:

$$L = \Phi \left| \frac{I}{B_0} \right|_{B_0=0}, \quad F_z = \pi \mu_0 \int_{C} \rho(l) J^2(l) n_z(l) dl$$

where $I$ is total current in the ring, found by integrating the surface current density.

To check the stability of the calculation and the convergence of the model, calculations were carried out with a doubling number of steps to determine the condition number $\nu$ of the matrix (9) in the infinity and Euclidean norms, the inductance of the ring and the force acting on the ring in the external field. The latter must be zero for a uniform field, so the deviation of the calculated value of the force from zero indicates the asymmetry of the found distribution of the superconducting current density. The results of the study are presented in Table I.

**TABLE I. TESTING THE MODEL FOR A RING WITH A CIRCULAR CROSS SECTION.**

<table>
<thead>
<tr>
<th>$N$</th>
<th>100</th>
<th>200</th>
<th>400</th>
<th>800</th>
<th>1600</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\nu_{\text{infinity}}$</td>
<td>281</td>
<td>568</td>
<td>1144</td>
<td>2295</td>
<td>4598</td>
</tr>
<tr>
<td>$\nu_{\text{Euclidean}}$</td>
<td>948</td>
<td>2688</td>
<td>7614</td>
<td>21550</td>
<td>60990</td>
</tr>
<tr>
<td>$L \cdot 10^7 \text{H}$</td>
<td>15.33187</td>
<td>15.35188</td>
<td>15.36189</td>
<td>15.36689</td>
<td>15.36939</td>
</tr>
<tr>
<td>$F \cdot 10^4 \text{dynes}$</td>
<td>9.21997</td>
<td>24.16275</td>
<td>-8.49803</td>
<td>-17.70065</td>
<td>10.07985</td>
</tr>
</tbody>
</table>

As can be seen from table I, the model is characterized by poor conditionality, increasing in proportion to the number of sampling steps. The convergence of the numerical solution is rather slow, but the calculation is stable, and the asymmetry of the solution is practically absent.
As a second example, we consider the problem of several bodies in the form of a system of two rings in a cylindrical screen. The geometrical parameters of the system were: the radii of the rings 28 mm and 40 mm, the radii of cross-sections of rings 10 mm and 20 mm, the base radius of the cylindrical screen 65 mm, height 120 mm, the corner radius of 10 mm, clearance between the rings 10 mm, gap between the bottom ring and the bottom of the screen 20 mm. The magnetic fluxes in the rings were equal to \(-1 \times 10^{-7}\) Wb on the upper ring and \(1 \times 10^{-7}\) Wb on the lower ring, respectively. The 100 sampling points on each ring and 200 points on the screen are used. The calculation results are presented in Figures 2 and 3.

In the system of three bodies, there are three forces, the sum of which, according to Newton's third law, must be equal to zero. The results of the calculation show that with an accuracy of 0.1% this equality is performed, which indicates the correctness of the solution of the model.

![Figure 2. Distribution of the superconducting current density on the upper (1) and lower (2) rings and on the cylindrical screen.](image_url)

![Figure 3. Distribution of the magnetic field in the gaps of two rings with a cylindrical screen.](image_url)
DISCUSSION

The developed method allows, in principle, to build models for a complex axisymmetric system of multi-connected superconducting bodies. Superconducting screens in the form of surfaces of rotation or planes perpendicular to the axis of symmetry of the system can also be considered. However, the models are not applicable in violation of the axial symmetry of the system. Asymmetric systems should be analyzed in more general models, such as those based on the finite element method [12].

At the same time, the models are characterized by poor conditionality, sharply increasing with increasing number of bodies, slow convergence proportional to the number of discretization points, and sensitivity to the non-uniformity of the discretization step on the ring cross-section contour. This leads to the need to use a large number of sampling points, especially on non-circular sections, which leads to an increase in the calculation time and demands on computational resources. However, by reducing the dimension of the problem when using the method of integral equations, the number of degrees of freedom, with the same accuracy of the calculation, is an order of magnitude less than in the finite element method.

Numerical calculations were carried out using a universal system of computer mathematics Maple 14.

REFERENCES