Stiffness Variation of a Cracked Rotor with a Semi-Elliptical Front

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Keywords: Stiffness variation, Strain energy release rate, Semi-elliptical crack.

Abstract. The purpose of this paper is to determine the variation of stiffness coefficients of a rotor containing a transverse crack with a semi-elliptical front. The crack front was considered to be straight in previous research efforts; however, the crack grows in an elliptical path when the rotor is subjected to cyclic loads, as determined through fatigue experiments and realistic cases. Therefore, considering the effect of a semi-elliptical surface crack, the dynamics of a semi-elliptical and straight cracked rotor are modelled using the finite element method, and the local flexibility introduced by the cracks are determined by the strain energy release rate (SERR). Finally, the comparative study of the stiffness variation for both types of cracked rotors exhibit the characteristics of crack breathing behaviour, and crack breathing has a significant influence on the variation of the direct and coupled stiffness.

Introduction

The existence and propagation of transverse fatigue surface cracks lead to the changes of the dynamic behaviours of rotors. Recently, many research studies [1] reveal that the growth of a crack on a rotating shaft under fatigue bending and/or tensile loads occurs in a semi-elliptical way. However, the crack front was usually assumed to be straight in the theory of strain energy release rate (SERR) [2], and this assumption is used to calculate the local compliance due to a crack. The crack front grows along an elliptical arc, with the crack propagation and the curvature of the arc declining gradually. At the beginning of the crack growth, the centre of the crack front is in the state of plane strain, and the stress intensity factors (SIFs) along the crack front are at their maximum values, but the intersection of the crack front and shaft surface is in the state of plane stress, and the SIFs are much smaller at that point. As a result, the crack growth rate at the centre of the crack front is larger than that at the intersection; therefore, the crack front develops along a curved path. Research studies [3], [4] have demonstrated that the surface cracks on a shaft grow along a semi-elliptical path. With the growth of the crack, the SIFs at the centre of the crack front gradually decrease along with the crack growth rate, but the SIFs at the intersection increase, and the crack growth rate rises there, as well. Consequently, the crack front becomes flat. The elliptical crack front is assumed to represent a constant crack length/crack depth ratio [4]. For an analysis of breathing behaviour and the stiffness variation of a cracked shaft, a semi-elliptical crack instead of a straight crack is taken into consideration in this paper.

Crack breathing, i.e., the periodic gradual opening and closing of the crack, and the consequent variation of the stiffness coefficient of the shaft is a unique phenomenon when a cracked rotor suffers weight and unbalanced loads. Papadopoulos [5] summarised the models of a breathing crack. Recently, Darpe [6] proposed the concept of the closure crack line (CCL), which separates the closed portion from the open portion of the crack by the position of the CCL. Additionally, Bachschmi [7] developed a 1D linear model to determine the boundary of the open and closed areas by the position of the main inertia axes of the crack surface. Both Darpe’s and Bachschmi’s model effectively reveal the so-called “crack closure effect”, that is, the crack remains closed as long as the stresses deriving from the external loads do not overcome the residual internal stresses [8]. However, in the discussion of the crack breathing and stiffness variation, the profile of the crack front was be assumed to be straight, and naturally, the curvature change of the crack front with the
crack depth was not considered. In fact, this assumption is not inconsistent with the actual situation.

In this paper, stiffness variation of a cracked rotor with a semi-elliptical surface crack is studied. First, the dynamical model of a semi-elliptical cracked rotor is modelled using the finite element method, and the local flexibility introduced by cracks are derived by SERR, considering the effect of the semi-elliptical surface crack. Then, a more accurate breathing behaviour is obtained by calculating the position of the CCL[6]. Finally, the stiffness variation is calculated by numerical simulation. The results of the semi-elliptical cracked rotor simulation are compared with those of the straight cracked rotor to indicate the influence of the semi-elliptical front and the curvature change of the crack front on the rotor’s dynamical characteristics.

The Stiffness of a Cracked Element

Local Flexibility Due To the Crack

A semi-elliptical cracked rotor segment is shown in Figure 1. The rotor segment is modelled by a Timoshenko beam element with six DOFs per node. The radius and length of the rotor segment are \( R \), \( l \), respectively, and the crack is at a distance \( x \) from the left end of the segment. The forces and moments loaded on the rotor segment are the axial forces \( P_1, P_2 \), the shear forces \( P_3, P_4, P_5, P_6 \), the torsional moments \( P_7, P_8 \) and the bending moments \( P_9, P_{10}, P_{11}, P_{12} \).

![Figure 1. A semi-elliptical cracked rotor segment.](image)

In considering the local flexibility due to the crack, the straight crack is replaced by a semi-elliptical crack. The geometric profile of a straight crack and a semi-elliptical crack are shown in Figure 2.

In the case of a straight crack in Figure 2 (a), the boundary \( b \) is

\[
b = \sqrt{R^2 - (R - a)^2}
\]

where \( R \) is the radius of the shaft, and \( a \) is the crack depth.
The height of the strip $h$ is expressed as

$$h = 2\sqrt{R^2 - \beta^2} \quad (2)$$

The crack depth $\alpha$ of this strip is

$$\alpha = \sqrt{R^2 - \beta^2} - (R - a) \quad (3)$$

The profile of the elliptical crack is assumed to exhibit a constant crack length/crack depth ratio, as illustrated in Figure 2 (b). The crack depth $a$ is obtained experimentally [3].

$$a = \frac{2s}{\pi} \quad (4)$$

where $s$ is the half-length of the crack profile and $s = \varphi R$, thus,

$$\varphi = \frac{\pi a}{2R} \quad (5)$$

The major semi-axis of the ellipse $b$ is calculated according to equation (6).

$$b = \frac{R \sin \varphi}{\sqrt{1 - \frac{R^2}{a^2} (1 - \cos \varphi)^2}} \quad (6)$$

The height of the strip $h$ is expressed as

$$h = 2\sqrt{R^2 - \beta^2} \quad (7)$$

From equations (4)-(7), the crack depth of the strip is obtained:

$$\alpha = \sqrt{R^2 - \beta^2} + \frac{a}{b} \sqrt{b^2 - \beta^2} - R \quad (8)$$

To obtain the flexibility coefficients due to the crack, the additional strain energy due to the crack based on the concepts of fracture mechanics can be expressed as

$$U^c = \int_A G(A) dA \quad (9)$$

Where $G(A)$ is the strain energy density function and is given by

$$G(A) = \frac{1}{E'} \left[ \left( \sum_{i=1}^{6} K_i^I \right)^2 + \left( \sum_{i=1}^{6} K_i^{II} \right)^2 \right] + \frac{1 + \nu}{E} \left[ \sum_{i=1}^{6} K_i^{III} \right] \quad (10)$$

Here, $E$ is Young’s modulus, $\nu$ is Poisson ration and $E' = E / (1 - \nu^2)$. $K_i^I$, $K_i^{II}$, and $K_i^{III}$ $(i = 1$ to $6)$ are the stress intensity factors (SIFs) corresponding to mode I, II and III of crack displacement, respectively. The stress intensity factors are given in reference [6].

Ultimately, the flexibility coefficients of the cracked element $g_{ij}^c$ are expressed as follows. The limits of integration of equation (11) range from $R \sin \varphi$ to $-R \sin \varphi$ for the semi-elliptical crack, rather than from $b$ to $-b$ for the straight crack, and $g_{11}^c = g_{22}^c = g_{33}^c = g_{12}^c = g_{13}^c = g_{23}^c = 0$.

$$g_{ij}^c = \frac{\partial^2 U^c}{\partial P_i \partial P_j} \quad (i, j = 1, 2, ..., 6) \quad (11)$$

**Total Stiffness Matrix**

The flexibility coefficients of the uncracked element $g_{ij}^0$ are derived using the strain energy approach [6], thus the total flexibility coefficients of a cracked element $\sigma$ are
\[ g_y = g_y^0 + g_y' = \frac{\partial^2 U^0}{\partial \rho \partial \rho} + \frac{\partial^2 U'}{\partial \rho \partial \rho} \]  

(12)

The flexibility matrix is

\[ [Fe] = [g_y] \text{det}6 \]  

(13)

The total stiffness matrix of a cracked element is \([k]'\), and \([Tr]\) is the transformation matrix [6].

\[ [k]' = [Tr][Fe]^{-1}[Tr]^T \]  

(14)

### Equations of Motion

The equations of motion are derived using the Timoshenko shaft element with two nodes and six degrees of freedom per node. Thus, the equations of motion in stationary coordinates considering the moment of inertia of the shaft and the shear distortion can be written as

\[ [M]^s \{\ddot{q}\} + ([Gy]^s + [C]^s)\{\dot{q}\}^s = \{f\}^s \]  

(15)

where \([M]^s\), \([Gy]^s\), \([C]^s\) and \([K]^s\) are the mass, damping, gyroscopic and stiffness matrices, respectively, for the system. \([\dot{q}]^s\), \([q]^s\) and \([\ddot{q}]^s\) are the acceleration, velocity and displacement vectors, respectively. The force vector \([f]^s\) contains gravity and the unbalanced excitation forces.

The Newmark method of direct integration of equations of motion was used to simulate the rotor’s response. The integration parameters are \(\alpha = 0.2525\) and \(\delta = 0.505\). The mass and the stiffness damping coefficients are \(m_y = 6.812\) and \(k_y = 3.311 \times 10^4\), respectively. The sampling frequency with this integration time step is \(f_s = 1024\) Hz. The first two natural frequencies of the bending vibrations of the rotor are 32.14 Hz and 139.92 Hz.

### Stiffness Variation

In this paper, the concept of the Closure Crack Line (CCL) proposed by Darpe [8] is used to model the crack breathing. To locate the CCLP, the crack edge is divided into 50 segments of equal length, as shown Figure 4. The sum of the SIFs (\(K^o\)) must be evaluated at any of these points along the crack front to determine the CCLP.
\[ K^0 = K_1^0 + K_2^0 + K_6^0 \]  

At each degree of rotation in a rotation, the CCLP is determined by the sign of \( K^0 \) in Eq. (16). A negative sign of \( K^0 \) indicates compressive stress occurs at that point; therefore, the crack is assumed to be closed there. Similarly, a positive sign of \( K^0 \) indicates tensile stress occurs, and the crack is open. Thus, the position along the crack edge where \( K^0 \) changes its sign is the CCLP.

To evaluate the stiffness variation, the crack is assumed to be located at the top of the shaft originally; therefore, the crack is completely closed due to the force of gravity. In stationary coordinates, the initial displacement \( \{q\}_0 \) is the static deflection of the uncracked rotor, and the initial velocity \( \{v\}_0 \) is zero. The stiffness matrix of the uncracked rotor \( \{K\}_0 \) is used as the initial one. After each degree of rotation, the stiffness matrix must be updated to account for the breathing behaviour of the crack. The process of estimating the response continues until the response becomes stable.

**Results and Discussion**

![Graphs showing variation of direct stiffness coefficients of SCR and SECR with rotation angle.](image)

The variation of the direct stiffness coefficients of the SCR and the SECR with rotation angle are presented in Figure 5 (a) ~ (f). It can be seen in figure 5 that all of the direct stiffness coefficients do not begin to decrease until the rotor has already rotated a certain angle instead of declining at the beginning of rotation. Next, the coefficients drop to a minimum value before the rotor rotates to 180°, and they return to the origin in a symmetrical way. Further, the angular region in which the stiffness does not change corresponds to the state that crack is fully closed or fully open. Consequently, all of the variations of the direct stiffness coefficients illustrated in figure 5 are consistent with the crack breathing behaviour.

In the case of the same crack depth, (e.g., \( \bar{a} = 0.3 \)), the variation of the direct stiffness coefficients of the SECR are compared with those of the SCR. \( k_{11} \) corresponding to the tensile force, \( k_{44} \) corresponds to the torsion force and \( k_{55} \) and \( k_{66} \), corresponding to the bending moment, are responsible for crack growth, so they are sensitive to the crack profile. The values of \( k_{11} \), \( k_{44} \), \( k_{55} \), \( k_{66} \) of the SCR in the regions of the crack being fully open are smaller than those of the SECR. This difference is observed because the crack opening area of the semi-elliptical crack is larger than that of the straight crack under the same crack depth. However, the \( k_{22} \) and \( k_{33} \) values exhibit only negligible differences for both cracked rotors because they are shear stiffness and are not sensitive to the crack profile.
The effect of crack depth on the variation of the direct stiffness coefficients are illustrated in Figure 5(a) ~ (f) simultaneously. It can be found that the variations of the direct stiffness of the SECR and SCR is not quite noticeable for very shallow depths (e.g., $\bar{a}=0.1$), but they clearly decrease with the increase of the crack depth, and the angular region over which the direct stiffness coefficients remain constant become narrow. The values of $k_{11}$, $k_{44}$, $k_{35}$, $k_{36}$ of the SECF are larger than those of the SCR at shallow depths, but they are the same at deeper depths (e.g., $\bar{a}=0.4$) because the semi-elliptical crack front becomes flat and tends to be straight. However, the shear stiffness coefficients $k_{22}$ and $k_{33}$ of both the SECR and the SCR are the same at a certain crack depth; therefore, they are not sensitive to the crack profile. In addition, the variation of $k_{22}$ and $k_{33}$ with crack depth are negligible for the SECR or the SCR, meaning that they are also not sensitive to the crack depth.

As with the discussion of the direct stiffness variation, the crack breathing also has an influence on the variation of the coupled stiffness coefficients. The variation of the coupled stiffness coefficients over a rotation for the SCR and the SECR are illustrated in Figure 6; however, in this case, the coupled stiffness coefficients $k_{15}$, $k_{16}$, $k_{34}$, $k_{35}$, $k_{36}$ and $k_{34}$ are negligible because they are near zero. In Figure 6, the angular region in which the stiffness coefficients remain constant correspond to the state that the crack is fully closed or fully open at the beginning and the end of the rotation. Those angular regions became narrow as the crack depth increased, i.e., the crack breathing is more noticeable for a deeper crack than for a shallow crack.

In the case of a crack depth of $\bar{a}=0.3$, the coupled stiffness coefficient between the longitudinal and bending $k_{36}$ change once, the longitudinal and bending $k_{34}$, and the shear and torsion $k_{24}$ change twice, and the shear and torsion $k_{34}$ change three times in a rotation. The values of $k_{34}$, $k_{36}$, and $k_{34}$ are different for the SECR and the SCR; therefore, they are sensitive to the crack profile. However, the shear and bending stiffness coefficients $k_{26}$ and $k_{35}$, respectively, are the same for the SCR and the SECR, which means that they are not sensitive to the crack profile.

As the crack depth increases, the values of $k_{35}$, $k_{16}$, $k_{24}$, $k_{34}$ clearly vary, but $k_{26}$ and $k_{35}$ vary only slightly at different crack depths; therefore, they are also not sensitive to the crack depth.

**Conclusions**

The direct stiffness coefficients $k_{11}$, $k_{44}$, $k_{35}$, $k_{36}$ are sensitive to the crack profile because they are responsible for crack growth. Their values for the SECF are larger than those for the SCR at
shallow depths, but they are the same at deeper depths. However, $k_{22}$ and $k_{33}$ are not only insensitive to the crack profile but also the crack depth for both the SECR and the SCR.

The values of the coupled stiffness coefficients $k_{15}$, $k_{16}$, $k_{24}$, $k_{34}$ are different at the same crack depth for the SECR and the SCR, i.e., they are sensitive to the crack profile. In addition, these coefficients clearly vary with increased crack depth. However, $k_{26}$ and $k_{35}$ are not only insensitive to the crack profile but also to the crack depth for the SECR and the SCR.

References


