Parametric Instability of Euler Beams under Random Wind Loads

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Abstract. The present study discusses the approach of dynamic stability analysis for pin-ended beams under random wind loads. The Schwarz’s inequality is adopted to determine the boundary of dynamic instability, and analyses under a real time history of wind forces are carefully carried out. Studies show that the relationship of variance and mean of wind load determines the state of parametric stability of the beam. Increase of the variance or the mean will make the beam be subject to instability. The beam becomes more stable with stronger elasticity, higher damping and density.

Introduction

The Euler beams, as common forms of members in modern structures, are significant research objects in the parametric instability field [1, 2]. Modern civil structures are constructed with greater height or span, resulting in beam members of very small natural frequencies, so wind-induced parametric resonance has gradually become a remarkable problem.

The dynamic instability of Euler beams has been studied by a lot of scholars [3-7]. However, in these previous studies, axial loads along the beams are commonly simplified as harmonic loads or step loads, whereas the wind load is arbitrary and random load and cannot be such simplified. Therefore, it is necessary to study the dynamic instability of Euler beams under random wind loads.

Methods

The studied Euler beam is shown in Figure 1. In the Figure, $x$ denotes the coordinate along the beam; $E$, $I$ and $l$ denote the Young’s module of material, the moment of inertia of section and the length, respectively, of the beam; $f(t)$ denote the random wind force axially acting on the beam. $f(t)$ can be expressed as:

$$f(t) = f_0 + g(t),$$

(1)

where $f_0$ is the mean component of wind force; $g(t)$ is the random fluctuating components with zero mean value.

Figure 1. The studied Euler beam under wind forces.

The Mathieu-Hill equations corresponding to normal modes of different orders have the same form as long as the parameters in the equation are calculated from the same order of mode, so the distributions of dynamic instability regions in different modes are identical while the parameters used for the region coordinates correspond to the same order. Consequently, the subscript of mode order in
the Mathieu-Hill equation can be omitted, and all parameters can be calculated by the first order of mode. Thus, the Mathieu-Hill equation corresponding to the system is:

$$\ddot{x} + 2\xi \dot{x} + \Omega^2 (1 - 2\mu g)x = 0,$$

(2)

where $x$ denotes the modal coordinate; $\xi$ is the modal damping coefficient; $\Omega$ is the natural frequency of the beam under $f_0$; $\mu$ is the excitation parameter.

To simplify Eq. 2, the following conversion formula is introduced:

$$x(t) = y(t) e^{-\xi t}.$$

(3)

Then we have:

$$\dot{y} + \left[ c + h(t) \right] y = 0,$$

(4)

where $c = \Omega^2 - \xi^2$ and $h(t) = -2\mu\xi^2 g(t)$.

Eq. 4 can be rewritten as:

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -(c + h) & 0 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}.$$  

(5)

Positive definite matrix $A$ is defined to obtain the module of $y$:

$$A = \begin{bmatrix} \alpha_1 & \alpha_2 \\ \alpha_2 & 1 \end{bmatrix}, \alpha_1 > 0, \alpha_1 - \alpha_2^2 > 0,$$

(6)

$$\|y\|_A = V = y^T A y.$$

(7)

The derivative of $V$ is

$$V = y^T B y,$$

(8)

$$B = \begin{bmatrix} -2\alpha_2 (c + h) & \alpha_1 - (c + h) \\ \alpha_1 - (c + h) & 2\alpha_2 \end{bmatrix}.$$  

(9)

Along a solution $y(t)$,

$$\frac{\dot{V}}{V} = \frac{y^T B y}{y^T A y} \leq \max_y \frac{Y^T B Y}{Y^T A Y},$$

(10)

$$\left( \frac{Y^T B Y}{Y^T A Y} \right)$$ is a Rayleigh quotient and hence

$$\max_y \frac{Y^T B Y}{Y^T A Y} = \lambda_{\max} \left( BA^{-1} \right),$$

(11)

where $\lambda_{\max}$ is the largest eigenvalue of $BA^{-1}$.

So we have:

$$\|y\|_A \leq \|y_0\|_A \exp \left\{ \frac{t}{2} E \left[ \lambda_{\max} \left( \dot{\lambda}_{\max} (t) \right) \right] \right\},$$

(12)

Therefore, for the system of Eq. 5 to be stable in norm $\| \cdot \|_A$, it’s required that
\[
\frac{1}{2} E\left[ \lambda_{\text{max}}(t) \right] < -\varepsilon, \quad \text{for any } \varepsilon > 0.
\]  

Hence, a sufficient condition for asymptotic stability of system of Eq. 2 is obtained:

\[
-2\xi + E\left[ \lambda_{\text{max}}(t) \right] < -\varepsilon.
\]

According to Eq. 6 and Eq. 9, we have:

\[
\lambda_{\text{max}} = \sqrt{\frac{(\alpha_1 - c)^2 + 4\alpha_2^2 + (4\alpha_2^2 - 2\alpha_1 + c)h + h^2}{\alpha_1 - \alpha_2^2}}.
\]

Substituting Eq. 15 into Eq. 14 results in:

\[
-2\xi + E\left[ \sqrt{\frac{(\alpha_1 - c)^2 + 4\alpha_2^2 + (4\alpha_2^2 - 2\alpha_1 + c)h + h^2}{\alpha_1 - \alpha_2^2}} \right] < -\varepsilon, \varepsilon > 0
\]

We can apply the Schwarz’s inequality to Eq. 16 and obtain a stability boundary

\[
E\left[ \frac{(\alpha_1 - c)^2 + 4\alpha_2^2 + (4\alpha_2^2 - 2\alpha_1 + c)h + h^2}{\alpha_1 - \alpha_2^2} \right] \leq 4\xi^2
\]

Substituting the contents of c and h into Eq. 17, we have

\[
\left( \alpha_1 - \Omega^2 + \xi^2 \right)^2 + 4\left( \Omega^2 - \xi^2 \right)\alpha_2^2 + 4\mu^2 \Omega^2 \sigma_g^2 - 4\xi^2 (\alpha_1 - \alpha_2^2) \leq 0
\]

where \(\sigma_g\) denotes the variance of \(g(t)\).

Solving the derivative of the left side of Eq. 18 on \(\alpha_1\) and \(\alpha_2\) results in a sufficient asymptotic stability boundary:

\[
\sigma_g^2 \frac{w^2}{4\xi^2 P_{cr}} + P_0 - P_{cr} = 0
\]

Analysis on Results

In order to verify the above approaches, an actual random wind field is gained from a wind tunnel test and converted into full scale. Mean component of the wind force is \(f_0=4.19e3 N\). The length, density and Young’s modulus of a steel beam are \(l=6 m\), \(\rho=7850 kg/m^3\) and \(E=206 GPa\). The area and inertia moment of section are \(I=1.35e-6 m^4\), \(A=1.56e-3 m^2\), respectively, and the mass of unit length is \(m=12.2 kg/m\). Therefore, the Euler critical load and natural frequency according to the 1st mode are \(P_{cr} = \pi^2 EI/l^2 = 7.6e4N\) and \(w = \pi^2 \sqrt{EI/m/l^2} = 41.3 Hz\), resulting in \(\Omega = w\sqrt{1-f_0/P_{cr}} = 40 Hz\) . The modal damping ratio is 0.005, so the damping coefficient \(\xi = \Omega \cdot 0.005 = 0.2\). Hence, the stability boundary can be determined by Eq. 19, as shown in Figure 2.
It’s obvious that the relationship of variance and mean of wind load decide the state of the beam for stability or instability. Increase of the variance or the mean will make the beam be more subject to instability. Therefore, when the mean wind force is fixed, the beams located at areas with higher turbulence intensity will be more possible to become unstable.

It can also be observed that, even the mean wind load is lower than the Euler buckling load, i.e. the beam is statically stable, but it will become dynamically unstable owing to the fluctuating wind load, taking \( P_0 = 4 \times 10^4 \)N and \( \sigma_g = 600 \)N for example.

Stability boundaries corresponding to 2 kinds of Young’s modulus, \( E_1 = 100 \)GPa, \( E_2 = 300 \)GPa, 2 kinds of modal damping ratio of 0.002 and 0.01, and 2 kinds of density, \( \rho_1 = 5500 \)kg/m\(^3\), \( \rho_2 = 9500 \)kg/m\(^3\) are shown in Figure. 3. It’s obvious that increment of all those parameters enlarge the stability regions, so the beam will be more stable with stronger elasticity, higher damping and larger mass.

Figure 2. The sufficient asymptotic stability boundary.

![Stability region](image)

Figure 3. The stability boundaries under different Young’s modulus, damping ratios and densities.
Conclusion

The relationship of variance and mean of wind load determines the state of parametric stability of the beam. Increase of the variance or the mean will make the beam be more subject to instability. When the mean wind force is fixed, the beams located at areas with higher turbulence intensity will be more possible to become unstable.

Even if the beam is statically stable under the mean wind load, it may become dynamically unstable owing to the fluctuating wind load. Increment of Young’s modulus, damping ratio, density make the beam be more stable.

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References