Counting Matching Numbers in Catacondensed Polyomino Systems

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Abstract. The matching counting problem has its own significance in mathematics and interconnection network of parallel computer system. Let $G$ be a graph, the total matching number is the total number of independent edge subsets in $G$. For general graphs, the matching counting problem has proven to be intractable and computing the total matching number is $\#P$ hard. This has led to an emphasis on studying this problem in particular classes of graphs. The polyomino system is a finite 2-connected plane graph such that each interior face (or say a cell) is surrounded by a regular square of length one. The catacondensed polyomino system is a chain polyomino system and its central line forms a tree. In this paper, the reduction formulas of computing the total matching number of any catacondensed polyomino system via three kinds of transfer matrices are obtained.

1 Introduction

All graphs considered here are finite, undirected and simple. Let $G$ be a graph. A matching of $G$ is a set of edges of $G$ which no two of them have common vertex. The total matching number (also called the number of monomer-dimer arrangements of $G$ in statistical mechanics or the Hosoya index of $G$ in mathematical chemistry), is defined as $Z(G) = \sum_{k=0}^{\infty} m(G,k)$, where $m(G,k)$ is defined to be the number of ways in which $k$ mutually independent edges can be selected in $G$ [1]. Counting all matchings of a graph is a general matching counting problem, which is not only physically intriguing but has a large variety of applications in chemistry and also, has its own significance in mathematics and interconnection network of parallel computer system [1-4]. For general graphs, the matching counting problem has proven to be intractable and computing the total matching number is $\#P$ hard. So, let's consider the specific graph. Polyominoes have a long and rich history, we convey for the origin polyominoes in [5]. A polyomino system is a finite two-connected plane graph so that each interior face (also called a cell) is surrounded by a regular square of length one. The catacondensed polyomino system is a chain polyomino system and its central line forms a tree. Up to now, the research on the catacondensed polyomino system is mostly focused on the path-like polyomino system, in which its central line forms a path [6-7]. In this paper, we consider the general catacondensed polyomino system. R. Cruz, C. A. Marín and J. Rada introduced the Hosoya vector of a graph at a given edge [8]. By this concept, we obtain some basic recurrence relations on the
total matching numbers of the path-like polyomino systems. Furthermore, the reduction formulas of computing the total matching number of any catacondensed polyomino system via three kinds of transfer matrices can be obtained.

2 Preliminary

Lemma 2.1. ([1, 2]) (1) Let $G$ be a graph consisting of components $G_1, G_2, \ldots, G_n$, then $Z(G) = Z(G_1)Z(G_2) \cdots Z(G_n)$. (2) Let $e = uv \in E(G)$. Then $Z(G) = Z(G - uv) + i(G - u - v)$.

By Lemma 2.1 we know that the matching number of the path with $n$ vertices is always a Fibonacci number such that $Z(P_n) = 1, Z(P_2) = 1, Z(P_3) = 2, \ldots$ [9].

Definition 2.1. ([8]) Let $uv$ be an edge of the graph $G$. We define the Hosoya vector of $G$ at $uv$, denote by $Z_{uv}(G)$, as the column vector $Z(G - u, Z(G - v), Z(G - u - v))^T$.

Note that $Z_{uv}(G) = Z(G - u, Z(G - v), Z(G - u - v))^T$.

By Lemma 2.1 we have

$$
E_{23} = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix},
$$

where $E_{23}$ is the permutation matrix.

Figure 1. Two graphs $G_1$ and $G_2$.

Theorem 2.1. Let $G$ be the graph obtained from the graph $H$ by attaching to it a square (also called a cell) at the edge $xy$ (see Fig.1), then $Z_{uv}(G) = Q_1 Z_{xy}(H)$ and $Z_{uv}(G) = Q_2 E_{23} Z_{xy}(H)$, where

$$
Q_1 = \begin{pmatrix}
2 & 1 & 1 & 1 \\
1 & 0 & 1 & 0 \\
1 & 1 & 0 & 0 \\
1 & 0 & 0 & 0
\end{pmatrix}
$$

and

$$
Q_2 = \begin{pmatrix}
2 & 1 & 1 & 1 \\
1 & 0 & 1 & 0 \\
0 & 2 & 0 & 1 \\
0 & 1 & 0 & 1
\end{pmatrix}.
$$

Proof. By Lemma 2.1 we have

$Z(G) = (2, 1, 1, 1) Z_{xy}(H)$, $Z(G - u) = (1, 0, 1, 0) Z_{xy}(H)$,$Z(G - v) = (1, 1, 0, 0) Z_{xy}(H)$, $Z(G - u - v) = (1, 0, 0, 0) Z_{xy}(H)$.

Then $Z_{uv}(G) = Q_1 Z_{xy}(H)$ holds by Definition 2.1. Similarly, we can get $Z_{uv}(G) = Q_2 E_{23} Z_{xy}(H)$ and $Z_{uv}(G) = Q_2 E_{23} Z_{xy}(H)$. The proof is complete.

Remark. By Theorem 2.1, we get $Q_1 X_0 = Q_2 E_{23} X_0$, where $X_0 = (2, 1, 1, 1)^T$ is the Hosoya vector of $P_2$. In particular, if $H$ is a linear polyomino chain with $n - 1$ cells, i.e. $H = L_{n-1}$, then $Z_{uv}(L_n) = Q_1 X_0$; and if $H$ is a zigzag polyomino chain with $n - 1$ cells, i.e. $H = Z_{n-1}$, then $Z_{uv}(Z_{n-1}) = Q_1 (Q_2 E_{23} Q_2)^{(n-2)/2} Q_2 X_0$ when $n$ is even, otherwise.
\[ i_w(Z_n) = Q_i(Q_2E_2Q_2)^{(n-1)/2} X_0. \]

3 Recurrence relations on catacondensed polyomino systems

**Theorem 3.1.** Let \( G_z \) be the graph obtained by attaching two graphs \( K, H \) to a cell at the edges \( uv \) and \( st \), respectively (see Fig. 1). Then \( Z_e(G_z) = Z_w'(K)Z_u'(H) \) and \( Z_w'(K) \) denotes the \((4 \times 4)\) matrix such that

\[
Z_w'(K) = \begin{cases} 
(E_kZ_u'(K))_k, & \text{if } e = us, \\
(E_{uv}E_{st}Z_u'(K))_k, & \text{if } e = vt,
\end{cases}
\]

where \( k = 1,2,3,4 \), \( E_1 = E^T \), \( E_k = E_{uv}E_{st}E_2 \),

\[
E_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad E_2 = \begin{pmatrix} 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 \end{pmatrix}.
\]

**Proof.** By Lemma 2.1 we have

\[
Z(G_z) = [(1 0 0 0)Z_u'(K)]Z(H) + [(0 1 0 0)Z_w'(K)]Z(H - s) + [(0 0 1 0)Z_w'(K)]Z(H - t) + [(0 0 0 1)Z_u'(K)]Z(H - t - s).
\]

Thus, \( (Z_u(G_z))_k = (E_kZ_{uv}(K))^T Z_{st}(H) \). The proofs of the cases \( k = 2,3,4 \) are similar. So, we get \( (Z_u(G_z))_k = (E_kZ_{uv}(K))^T Z_{st}(H) \) \( (k = 1,2,3,4) \), where \( (Z_u(G_z))_k \) denotes the \( k \)-th element in \( Z_u(G_z) \). Furthermore, denote by \( Z_w'(K) \) the \((4 \times 4)\) matrix such that

\[
Z_w'(K) = ((E_kZ_{uv}(K))^T)_k, \quad \text{where} \quad k = 1,2,3,4.
\]

Then we obtain \( Z_w'(G_z) = Z_w'(K)Z_u'(H) \). For \( e = vt \), the proof is similar.

![Fig. 2](image)

Two types of \( G_3 \).

**Theorem 3.2.** Let \( G_3 \) be the graph obtained by attaching two graphs \( K, H \) to a \( L_2 \) at the edges \( xy \) and \( st \), respectively (see Fig. 2, there are two attaching types: A-type and B-type).

1. For A-type, we have \( Z_w(G_3) = Z_{xy}(K)Z_{st}(H) \), where \( Z_{xy}(K) \) denotes the \((4 \times 4)\) matrix such that \( Z_{xy}(K) = (T_kZ_{xy}(K))^T \) \( (k = 1,2,3,4) \),

\[
T_1 = \begin{pmatrix} 2 & 1 & 0 & 0 \\
1 & 1 & 0 & 0 \\
1 & 0 & 2 & 1 \\
1 & 0 & 1 & 1 \end{pmatrix}, \quad T_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 \end{pmatrix}, \quad T_3 = \begin{pmatrix} 1 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 \\
1 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 \end{pmatrix}, \quad \text{and} \quad T_4 = \begin{pmatrix} 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \end{pmatrix}.
\]

2. For B-type, we have \( Z_w(G_3) = Z_{xy}(K)Z_{xy}(H) \), where \( Z_{xy}(K) \) denotes the \((4 \times 4)\) matrix such that \( Z_{xy}(K) = (E_kT_kE_{st}Z_{xy}(K))^T \) \( (k = 1,2,3,4) \).
Proof. (1) For A-type of $G_3$ in Fig. 2, by Lemma 2.1 we have

$$Z(G_3) = (2 \ 1 \ 0 \ 0)Z_{uv}(K)Z(H) + (1 \ 1 \ 0 \ 0)Z_{uv}(K)Z(H - s)$$

$$+ (1 \ 0 \ 2 \ 1)Z_{v}(K)Z(H - t) + (1 \ 0 \ 1 \ 1)Z_{v}(K)Z(H - s - t).$$

Thus, $(Z_{w}(G_3))_1 = (T_{k}Z_{v}(K))^TZ_{w}(H)$. For the cases $k = 2,3,4$ the proofs are similar. Then, we get $(Z_{w}(G_3))_k = (T_{k}Z_{v}(K))^TZ_{w}(H)$ $(k = 1,2,3,4)$, where $(Z_{w}(G_3))_k$ denotes the $k$-th element in $Z_{w}(G_3)$. Furthermore, denote by $Z_{v}^{*}(K)$ the $(4 \times 4)$ matrix such that

$$Z_{v}^{*}(K) = (T_{k}Z_{v}(K))^T_k$$

where $k = 1,2,3,4$.

So we have $Z_{w}(G_3) = Z_{w}^{*}(K)Z_{w}(H)$. Similarly, we can prove (2). The proof is complete.

By the same way, we can get the following Theorem 3.2:

**Theorem 3.2.** Let $G_3$ be the graph obtained by attaching two graphs $K$, $H$ to a $L_3$ at the edges $xy$ and $st$, respectively (see Fig. 2). (1) For A-type, we have $Z_{w}(G_3) = Z_{w}^{*}(K)Z_{w}(H)$, where $Z_{v}^{*}(K)$ denotes the $(4 \times 4)$ matrix such that $Z_{v}^{*}(K) = (T_{k}Z_{v}(K))^T_k$ $(k = 1,2,3,4)$, $T_1 = T_2$, $T_2 = T_3$.

$$T_1 = \begin{pmatrix}
0 & 0 & 0 & 0 \\
2 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 \\
1 & 0 & 2 & 1
\end{pmatrix}$$

and $T_2 = \begin{pmatrix}
0 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 \\
1 & 0 & 1 & 1
\end{pmatrix}$.

For B-type, we have $Z_{w}(G_3) = Z_{w}^{*}(K)Z_{w}(H)$, where $Z_{v}^{*}(K)$ denotes the $(4 \times 4)$ matrix such that $Z_{v}^{*}(K) = (E_{k}T_{k}Z_{v}(K))^T_k$ $(k = 1,2,3,4)$.

**4 Examples on catacondensed polyomino systems**

Let $e = uv$ be an edge of the graph $G$. Denote by $\text{deg}(v)$ the degree of the vertex $v$ in $G$. If $\text{deg}(u) = \text{deg}(v) = 2$ then the edge $e$ is called the $2-2$ degree edge of $G$. If $\text{deg}(u) = 2$, $\text{deg}(v) = 3$ (or $\text{deg}(u) = 3$, $\text{deg}(v) = 2$) then the edge $e$ is called the $2-3$ degree edge of $G$.

In what follows, we consider the catacondensed polyomino systems with Y-branch at a cell:

**Example 1.** Let the graph $G_i = G_i(\text{us})L_j$ be obtained by attaching a linear polyomino chain $L_j$ with $j$ cells to the graph $G_i$ (see Fig. 1) at the edge $\text{us}$, by Theorems 2.1 and 3.1 we get

$$Z_e(G_i) = Q(\text{us})Z_{\text{us}}(K)Z_{\text{us}}(H),$$

where $e$ is a $2-2$ degree edge of $L_j$ in $G_i$, $Z_{\text{us}}(K) = (E_{k}Z_{\text{us}}(K))^T_k$ $(k = 1,2,3,4)$.

**Example 2.** Let the graph $G_i = G_i(\text{us})Z_j$ be obtained by attaching a corner $Z_j$ to the graph $G_i$ (see Fig. 1) at the edge $\text{us}$, by Theorems 2.1 and 3.1 we have

$$Z_e(G_i) = Q(\text{us})Q(\text{us})Z_{\text{us}}(K)Z_{\text{us}}(H)$$

or $Z_e(G_i) = Q(\text{us})Q(\text{us})Z_{\text{us}}(K)Z_{\text{us}}(H)$, where $e$ is a $2-2$ degree edge of $Z_j$ in $G_i$, $Z_{\text{us}}(K) = (E_{k}Z_{\text{us}}(K))^T_k$ $(k = 1,2,3,4)$.

**Example 3.** Let the graph $G_i$ be obtained from the graphs $G_i$ and $W$ by identifying the edge $e_w$ of $W$ to a $2-2$ degree edge of $Z_i$ in $G_i$, by Theorems 3.1 and 3.2, we know that

$$Z_e(G_i) = Z_{w}(G_i)Q(\text{us})Z_{\text{us}}(W),$$

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where $e$ is a 2-3 degree edge of $Z_i$ in $G_i$, 
\[
Z^*_w(G_k) = (S_kZ^*_w(K)Z^*_w(H))^T_k \quad (k = 1, 2, 3, 4)
\]
or
\[
Z^*_w(G_k) = (E_{23}S_kE_{23}Z^*_w(K)Z^*_w(H))^T_k \quad (k = 1, 2, 3, 4),
\]
where $S_k = T_k$ or $T'_k$.

For the catacondensed polyomino system with four branches on one cell, we can get the reduction formulas of computing its matching number through similar discussions.

5 Conclusions

The polyomino system, like hexagonal system, is a research hotspot in statistical mechanics and mathematical chemistry. This paper mainly studies the general catacondensed polyomino system, in which its central line forms a tree and each cell can have at most four adjacent cells. By the Hosoya vector, some recurrence relations on the total matching numbers of the path-like polyomino systems are given (see Theorems 2.1, 3.1 and 3.2). These relationships only depend on three types of transfer matrices ($Q$, $E$, and $T_k$), from which the reduction formulas of computing the total matching number of any catacondensed polyomino system via transfer matrices can be obtained.

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References

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