Locating-Total Domination Number in Strong Product of Two Paths

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Abstract. In a monitoring system, each node's status is unique, and the system can accurately locate the node when there is a problem, which can be modeled through graphs’ locating-total domination. When laying out the monitoring system, it is crucial to select the root node. Given a graph $G$, its locating-total domination number is the minimum cardinality of $G$'s locating-total dominating set. In this paper, we give the bounds of this number for the strong product of two paths.

Introduction

Considering the problem of monitor placement in a monitoring system, each node (consisting of the monitoring device) is linked to the monitor and is modeled by graphs’ total-domination. If someone goes wrong in the system, it will be located by the monitor uniquely, which can be simulated by the graph’s locating-total domination. When laying out the monitoring system, we need to consider many factors, such as the cost and the work efficiency of the system, how to select the root node is crucial.

The study on graphs’ locating-dominating set of was pioneered by Slater [1-3], which has been extended to total domination. Locating-total domination in graphs was firstly studied by Haynes et al.[4], which has been further studied in References [5-10].

Given a simple connected graph $G$ with a vertex set $V(G)$ and an edge set $E(G)$. Denote the open neighborhood of a vertex $v$ by $N(v)$, which is the set of vertices linking to it. Denote the closed neighborhood of $v$ by $N[v]$, which is $\{v\} \cup N(v)$. Given a subset $S \subseteq V(G)$, we defined the open neighborhood of $S$ as $N(S)=\bigcup_{v \in S} N(v)$, and defined the closed neighborhood as $N(S) = \bigcup_{v \in S} N[v]$. Given an $S$, if every vertex in $V(G)$ is adjacent to at least one of its vertex, then $S$ is a total dominating set. Such $S$ is a locating-total dominating set if $N(u) \cap S \neq N(v) \cap S$. Here, this set is abbreviated as LTD-set. A graph with no isolated vertex contains a LTD-set. Denoted the locating-total domination number of $G$ by $r^t_L(G)$. It is the minimum cardinality of $G$’s locating-total dominating set.

The strong product of two simple graphs is mentioned in References[11,12]. Given two simple connected graphs $G$ and $H$, we denoted their strong product by $G \boxtimes H$. Two vertices $(g_i,h_j)$ and $(g_j,h_i)$ of $V(G \boxtimes H)$ connect to each other if and only if they satisfy one of the following conditions:

1. $g_i = g_i$ and $h_j, h_j \in E(H)$,
2. $g_i, g_j \in E(G)$ and $h_i = h_i$,
3. $g_i, g_j \in E(G)$ and $h_i h_j \in E(H)$.

Let $S$ be a subset of $V(G)$. All the edges of $G$ with all endpoints in $S$ consists of the edge set. This obtained subgraph of $G$ is called the derived subgraph of $S$ in $G$. Let $B$ be a subset of $E(G)$, and its vertex set is composed of all the vertices of $G$ that are associated with at least one edge in $B$. Taking $B$ as the edge set, the derived subgraph of $G$ is called the edge derived subgraph of $B$ in $G$.

Denote a path with $n$ vertices by $P_n$. The strong product of two paths $P_m$ and $P_n$ is denoted by $P_m \boxtimes P_n$. In [2], Haynes and Henning showed that $r^t_L(P_n) = \left\lceil \frac{n}{2} \right\rceil + \frac{1}{4} - \left\lceil \frac{1}{4} \right\rceil$.
We derived the bounds for the locating-total domination number of $P_m \square P_n$. We use $v_{ij}$ to denote the vertex in $V(P_m \square P_n)$, where $1 \leq i \leq m$, and $1 \leq j \leq n$. Let $S$ be a LTD-set of $P_m \square P_n$. Define that $S_j = S \cap \{v_{ij}, v_{i+1,j}, \ldots, v_{ij+m}\}$, and $|S_j|$ is the vertex number of $S_j$. Besides, $EC$ refers to the columns which exclusive vertex in $S$.

**Locating-Total Domination Number of $P_2 \square P_n$**

We calculated the locating-total domination number of $P_2 \square P_n$.

**Theorem 1.** If $n \geq 3$, $\gamma_L^t(P_2 \square P_n) = n$.

**Proof.** Let $S = \{v_{ij} \mid i = 1,3,5,\ldots \} \cup \{v_{2j} \mid j = 2,4,6,\ldots\}$ (see Fig.1).

One can check that $S$ is a LTD-set of $P_2 \square P_n$, and $|S| = n$. Thus $\gamma_L^t(P_2 \square P_n) \leq n$.

![Figure 1. A LTD-set of $P_2 \square P_n$.](image)

Assume that $\gamma_L^t(P_2 \square P_n) < n$, there is $|S_i| = 0$ for some $i$ with $1 \leq i \leq n$. Since $N[v_{ij}] = N[v_{ij+k}]$ and $v_{ij}, v_{ij+k} \notin S$, then $N(v_{ij}) \cap S = N(v_{ij+k}) \cap S$, it is inconsistent with the definition of LTD-set, so $\gamma_L^t(P_2 \square P_n) \geq n$.

Based on the analysis above, $\gamma_L^t(P_2 \square P_n) = n$.

**Locating-Total Domination Number of $P_3 \square P_n$**

We calculated the locating-total domination number of $P_3 \square P_n$.

**Theorem 2.** If $n \geq 3$, $\gamma_L^t(P_3 \square P_n) = n - \left\lfloor \frac{n}{2} \right\rfloor$.

**Proof.** If $n$ is odd, $S = \{v_{1i} \mid i = 2,4,6,\ldots,n - 1\} \cup \{v_{2j} \mid j = 1,3,5,\ldots,n\}$.

If $n$ is even, $S = \{v_{12}, v_{2j}\} \cup \{v_{2i} \mid i = 3,5,7,\ldots,n-1\}$ (see Fig.2).

It is not difficult to check that $S$ is a LTD-set of $P_3 \square P_n$, and $|S| = n$. Thus $\gamma_L^t(P_3 \square P_n) \leq n$.

![Figure 2. A LTD-set of $P_3 \square P_n$.](image)

Let $S$ be a LTD-set of $P_3 \square P_n$. Therefore, $|S_{i-1}| = |S_i| = |S_{i+1}| = 0$, where $2 \leq i \leq n-1$. When $|S_i| = |S_{i+1}| = 0$, we break up the graph between column $i$ and column $i+1$, then columns $i$ and $i+1$ are viewed as boundary columns. The graph $P_3 \square P_n$ is divided into several parts, and none of them contain $|S_i| = |S_{i+1}| = 0$. Those parts can be viewed as new graphs.

**Conjecture 1:** When $|S_i| = 0(2 \leq i \leq n-1)$, then there exist positive integers $p$ and $q$, such that $|S_{i-p}| \geq 2$, $|S_{i+q}| \geq 2$, and $|S_k| = 0(0 \leq i-p \leq k \leq i+q \leq n, k \neq i)$.
The purpose is to allow columns that do not contain vertex in $S$ can borrow vertex in $S$ from columns that contain two or more vertex in $S$. The proof is completed by considering the three cases:

**Case 1.** $|S_i| = 0$, $|S_{i-1}| = 1$, $|S_{i+1}| = 1$.

All the situations have been listed in Fig.3. By the definition of LTD-set, it is easy to prove that those are not true except situations in Fig.2c and Fig.2g. Without loss of generality, we focus on the right side of Fig.2c (see Fig.4).

In Fig.4, if $|S_{i+2}| \geq 2$, the conjecture holds. By the definition of LTD-set, we take $|S_{i+2}| = 1$, it clear that $v_{2(i+2)} \in S$. Since $v_{i(i+1)} \cap S \neq v_{2(i+2)} \cap S$ and $v_{2(i+2)} \cap S = v_{3(i+2)} \cap S$, we get $|S_{i+3}| \geq 1$ and $v_{2(i+3)} \in S$. If $|S_{i+3}| \geq 2$, the conjecture holds. Otherwise, $|S_{i+3}| = 1$, and $S_{i+3} = \{v_{2(i+3)}\}$. Since $v_{1(i+2)} \cap S \neq v_{1(i+3)} \cap S \neq v_{3(i+3)} \cap S$, we can get $|S_{i+4}| \geq 2$.

Obviously, for case 1, the conjecture holds.

**Case 2.** $|S_i| = 0$, $|S_{i-1}| = 3$, $|S_{i+1}| = 1$, or $|S_i| = 0$, $|S_{i-1}| = 1$, $|S_{i+1}| = 3$.

The column $i$ can borrow a vertex in $S$ from column $i-1$ or column $i+1$, it does not affect the integrity of the conjecture.

**Case 3.** $|S_i| = 0$, $|S_{i-1}| = 2$, $|S_{i+1}| = 1$, or $|S_i| = 0$, $|S_{i-1}| = 1$, $|S_{i+1}| = 2$.

Without loss of generality, we focus on $|S_i| = 0$, $|S_{i-1}| = 2$, $|S_{i+1}| = 1$.

All the situations have been listed in Fig.5, it is not difficult to check that situations in Fig.5a,5b,5c,5d are unreasonable by considering the vertices in column $i$. Situations in Fig.5e,5f,5g,5h can be proved the same way as the situation in Fig.4.

For the situation shown in Fig.5i, it is clear that $|S_{i+2}| \geq 1$. We assumed that $S_{i+2} = \{v_{i(i+2)}\}$. Since $v_{3(i+1)} \cap S \neq v_{3(i+2)} \cap S$, we get $|S_{i+3}| \geq 1$. If $|S_{i+3}| = 1$, $S_{i+3} = \{v_{2(i+3)}\}$. When it appears another EC(column $i+k$) closest to column $i$ from the right side, than $S_{i+k} = 1(1 < x < k)$. From the proof of case 1 and the analysis of Fig.5j, we can get $S_{i+k-1} = \{v_{2(i+k-1)}\}$ and $k \geq 4$. 

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Since \( |S_{i+1}| = |S_{i+2}| = \cdots = |S_{i+k-1}| = 1 \), \( N(v_{1i}) \cap S_{i+1} = N(v_{2i}) \cap S_{i+1} = N(v_{3i}) \cap S_{i+1} \), and \( N(v_{1(i+k)}) \cap S_{i+k-1} = N(v_{2(i+k)}) \cap S_{i+k-1} = N(v_{3(i+k)}) \cap S_{i+k-1} \), the column \( i \) and the column \( i+k \) can be viewed as boundary columns that split off the column from \( i+1 \) to \( i+k-1 \).

Based on the analysis above, \( G \) is divided into as many parts as possible. All the EC in \( G \) are either boundary column of these parts or can borrow a vertex in \( S \) from columns that contain two or more vertices in \( S \). For each part, \( n' \) is defined as the number of columns, \( S' \) is this subgraph’s locating-total dominating set. Obviously, only if both boundaries of this subgraph are EC and there is another EC in the subgraph, that \( |S'| \) can be less than \( n' \) and one EC cannot borrow vertex in \( S \) from the left or right sides. Besides, it is not difficult to prove that if \( |S_1| = 0 \), then \( |S_2| \geq 2 \), and if \( |S_2| = 2 \), then \( S_2 = \{v_{12}, v_{32}\} \). When \( |S_n| = 0 \), the same as \( |S_1| = 0 \). In this case, \( |S'| \geq n' - 1 \). Now we want to find the minimum value of \( n' \). In Fig.6, it is clear that \( S_3 = \{v_{23}\} \) and \( S_7 = \{v_{27}\} \). Base on the proof in case 1, when both boundary of this subgraph are EC and there exist another EC, it is easy to prove that \( n' \geq 9 \). In Fig.7, we show the case when \( n' = 10 \) and \( n' = 11 \). As the subgraph in Fig.6 and Fig.7, several subgraphs of \( P_3 \times P_3 \) can be added (see Fig.7c). Therefore, when \( 9 \leq n' \leq 17 \), it can always take \( |S'| = n' - 1 \).

Based on the analysis above, we divide \( G \) as many as possible into the subgraph which \( n' = 9 \), therefore, \( r_1(P_3 \times P_n) = n - \left\lfloor \frac{n}{9} \right\rfloor \).

**Figure 5. Situations in case 2.**

**Figure 6.** A LTD-set of \( P_3 \times P_n \).

**Figure 7.** \( n' \) could be 10 to 17, such that \( |S'| = n' - 1 \).

**Locating-Total Domination Number of \( P_4 \times P_n \)**

We calculated the bounds of the locating-total domination number of \( P_4 \times P_n \).
Theorem 3. If $n \geq 4$, $n+1 \leq r^L(P_4 \Box P_n) \leq n + 2$.

Proof. If $n$ is odd, $S = \{v_{i1}, v_{in}\} \cup \{v_{2i} | i = 1, 3, 5, \cdots, n\} \cup \{v_{3j} | j = 2, 4, 6, \cdots, n-1\}$.

If $n$ is even, $S = \{v_{i1}, v_{in}\} \cup \{v_{2i} | i = 1, 3, 5, \cdots, n-1\} \cup \{v_{3j} | j = 2, 4, 6, \cdots, n\}$ (see Fig.8).

One can find that $S$ is a LTD-set of $P_4 \Box P_n$, and $|S| = n + 2$. Obviously, $r^L(P_4 \Box P_n) \leq n + 2$.

![Figure 8. A LTD-set of $P_4 \Box P_n$.](image)

Now we discuss the lower bound of $r^L(P_4 \Box P_n)$. Let $S$ be $P_4 \Box P_n$’s LTD-set. There is no $i$ such that $|S_{i-1}| = |S_i| = |S_{i+1}| = 0$. For $|S_i| = |S_{i+1}| = 0$, we break up the graph between columns $i$ and $i+1$, which can be both viewed as boundary columns of their respective parts.

When $|S_i| = 0$, it easy to prove that $|S_{i-1}|$ and $|S_{i+1}|$ cannot both equal to 1 with $2 \leq i \leq n-1$. In fact, conjecture 1 can be used here as well. We proved the conjecture the two cases:

**Case 1.** $|S_i| = 0$, $|S_{i-1}| = 2$, $|S_{i+1}| = 1$, or $|S_i| = 0$, $|S_{i-1}| = 1$, $|S_{i+1}| = 2$.

Without loss of generality, we focus on $|S_i| = 0$, $|S_{i-1}| = 2$, $|S_{i+1}| = 1$.

All the situations have been list in Fig.9. It’s easy to prove that situations in Fig.9a,9b,9d,9g,9h,9i,9j do not conform to the definition of LTD-set by considering the neighbor of the gray vertices. For situations in Fig.9c,9f, if $|S_{i+2}| \leq 1$, it cannot be true by considering the neighbor of the gray vertices(see Fig.10a,10b). Therefore, $|S_{i+2}| \geq 2$. For the situation in Fig.9e, we have $|S_{i+2}| \geq 1$, if $|S_{i+2}| = 1$, then $v_{5(i+1)} \in S$, it is clear that $|S_{i+1}| \geq 2$ (see Fig.10c). Therefore, the conjecture holds.

![Figure 9. Situations in case 1.](image)

![Figure 10. The proof process of case 1.](image)
Case 2. \( |S_i| = 0, \ |S_{i-1}| = 3, \ |S_{i+1}| = 1, \) or \( |S_i| = 0, \ |S_{i-1}| = 1, \ |S_{i+1}| = 3 \).

The column \( i \) can borrow a vertex in \( S \) from column \( i-1 \) or column \( i+1 \), it does not affect the integrity of the conjecture.

When \( |S_i| = 0 \) or \( |S_n| = 0 \), it is not difficult to prove that \( |S_2| \geq 3 \) or \( |S_{n-1}| \geq 3 \).

Therefore, once the graph appears \( EC \), column which contains three or more vertices in \( S \) that appear on the left or right sides, or columns which contain two or more vertices in \( S \) that appear on the left and right sides. In this way, the locating-total domination number of each part is at least one more than its column number, thus is \( |S| \geq n + 1 \).

If there is no \( EC \) in \( G \). When \( |S_1| = 1 \) and \( |S_2| = 1 \), there are only two cases, that are \( \{v_{21}, v_{32}\} \in S \) or \( \{v_{31}, v_{22}\} \in S \), by the definition of LTD-set, there must be \( |S_i| \geq 2 \). When \( |S_n| = 1 \) and \( |S_{n-1}| = 1 \), the situation is similar. Therefore, when \( |S_i| \geq 1 \) with \( 1 \leq i \leq n \), there must be some column where \( |S_i| \geq 2 \). Thus \( |S| \geq n + 1 \).

Based on the analysis above, we have \( n + 1 \leq r^t(P_n \boxtimes P_n) \leq n + 2 \).

**Locating-Total Domination Number of** \( P_m \boxtimes P_n \)

We calculated the bounds of the locating-total domination number of \( P_m \boxtimes P_n (n \geq m \geq 5) \).

Let \( m = 4k + x \), where \( x \) and \( k \) are integers, and \( k \geq 1, 0 \leq x \leq 3 \), and \( n \equiv a \pmod{2} \). Let \( S \) be a LTD-set of \( P_m \boxtimes P_n \).

According to the Fig.11, we define \( A \) as a set of vertices.

When \( n \) is odd, \( A = \{v_{(4t)} \mid i = 1, 3, 5, \ldots, n\} \cup \{v_{(4t-3i)} \mid j = 2, 4, 6, \ldots, n-1\} (1 \leq t \leq k) \).

When \( n \) is even, \( A = \{v_{(4t)} \mid i = 1, 3, 5, \ldots, n-1\} \cup \{v_{(4t-3i)} \mid j = 2, 4, 6, \ldots, n\} (1 \leq t \leq k) \).

In both cases, it is easy to figure out that \( |A| = \left\lfloor \frac{n}{4} \right\rfloor (n + 2) \). The proof is completed by discussing the four cases:

Case 1. If \( x = 0 \), let \( S = A \), than \( |S| \geq |A| = \left\lfloor \frac{n}{4} \right\rfloor (n + 2) \).

Case 2. If \( x = 1 \), let \( S = A \cup B \), \( B \) is a LTD-set of \( P_{n-a} \), than \( |S| \geq |A| + |B| = \left\lfloor \frac{n}{4} \right\rfloor (n + 2) + \left\lfloor \frac{n-2a}{2} \right\rfloor \).

Case 3. If \( x = 2 \), let \( S = A \cup C \), \( C \) is a LTD-set of \( P_n \boxtimes P_{n-a} \), than \( |S| \geq |A| + |C| = \left\lfloor \frac{n}{4} \right\rfloor (n + 2) + n - 2a + \left\lfloor \frac{n-a}{2} \right\rfloor \).

Case 4. If \( x = 3 \), let \( S = A \cup D \), \( D \) is a LTD-set of \( P_{3} \boxtimes P_n \), than \( |S| \geq |A| + |D| = \left\lfloor \frac{n}{4} \right\rfloor (n + 2) + n - \left\lfloor \frac{n}{2} \right\rfloor \).

One can find that \( S \) is their own locating-total dominating set. Therefore,
\[
\begin{align*}
\tau_t^t(P_m \boxtimes P_n) & \leq \left\lfloor \frac{n+2}{4} \right\rfloor (m+2) & m \equiv 0 \pmod{4} \\
\tau_t^t(P_m \boxtimes P_n) & \leq \left\lfloor \frac{n+2}{4} \right\rfloor (m+2) + \left\lfloor \frac{n-2+a}{4} \right\rfloor - \left\lfloor \frac{n-2+a}{4} \right\rfloor & m \equiv 1 \pmod{4}, n \equiv a \pmod{2} \\
\tau_t^t(P_m \boxtimes P_n) & \leq \left\lfloor \frac{n+2}{4} \right\rfloor (n+2) + n - 2 + a & m \equiv 2 \pmod{4}, n \equiv a \pmod{2} \\
\tau_t^t(P_m \boxtimes P_n) & \leq \left\lfloor \frac{n+2}{4} \right\rfloor (n+2) + n - \left\lfloor \frac{n}{4} \right\rfloor & m \equiv 3 \pmod{4}
\end{align*}
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References