The Number of Maximal Independent Sets in the $P_m$-extended of Graphs

Jenq-Jong Lin and Min-Jen Jou*
Ling Tung University, Taichung 40852, Taiwan
*Corresponding author

Keywords: Star, Path, Maximal independent set, $P_m$-extended.

Abstract. A maximal independent set is an independent set that is not a proper subset of any other independent set. Given a graph $G$ of order $n$, we define the $P_m$-extended of $G$, denoted $G(m)$, as the graph consisting of $G$ together with $n$ copies of $P_m$, a leaf of each path attached to exactly one vertex of $G$. In this paper, we determine the number of maximal independent sets of the $P_m$-extended of a star and a path.

Introduction and Preliminaries

Let $G=(V,E)$ be a simple undirected graph. A subset $I \subseteq V$ is independent if there is no edge of $G$ between any two vertices of $I$. A maximal independent set is an independent set that is not a proper subset of any other independent set. The set of all maximal independent sets of $G$ is denoted by $MI(G)$ and its cardinality by $mi(G)$. For a vertex $x \in V(G)$, let $MI_{-x}(G)=\{I \in MI(G) : x \notin I\}$ and $MI_{+x}(G)=\{I \in MI(G) : x \in I\}$. The cardinalities of $MI_{+x}(G)$ and $MI_{-x}(G)$ are denoted by $mi_{+x}(G)$ and $mi_{-x}(G)$, respectively. Trivially, $mi(G)=mi_{+x}(G)+mi_{-x}(G)$. For a set $A \subseteq V(G)$, the deletion of $A$ from $G$ is the graph $G-A$ obtained from $G$ by removing all vertices in $A$ and their incident edges. If $A=\{v\}$ is a singleton, we write $G-v$ rather than $G-\{v\}$. Denote by $S_m$ (respectively, $P_m$) a star (respectively, path) with $m$ vertices. Given a graph $G$ with $n$ vertices, we define the $P_m$-extended of $G$, denoted $G(m)$, as the graph consisting of $G$ together with $n$ copies of $P_m$, a leaf of each path attached to exactly one vertex of $G$. Throughout this paper, for simplicity, let $p_m=mi(P_m)$ and $G(m)=mi(G(m))$.

In 1972, Karp [4] has shown that the independent set problem is NP-complete. Although the problem of finding the size of a maximal independent set of a graph has been extensively studied (see[1], [5], [6]), there are few works about counting the number of maximal independent sets. The purpose of this paper is to evaluate the number of maximal independent sets in the $P_m$-extended of a star and a path.

Lemma 1. ([3]) If $G$ is the union of two disjoint graphs $G_1$ and $G_2$, then $mi(G)=mi(G_1) \cdot mi(G_2)$.

Lemma 2. $p_0=1$, $p_1=1$, $p_2=2$ and $p_{m+3}=p_m+p_{m+1}$, $m \geq 1$.

Proof. It is easy to check that $p_0=mi(P_0)=1$, $p_1=mi(P_1)=1$, $p_2=mi(P_2)=2$. In addition, let $u$ be a leaf of $P_{m+3}$, then $p_{m+3}=mi(P_{m+3})=mi_u(P_{m+3})+mi_v(P_{m+3})=mi(P_m)+mi(P_{m+3})=p_m+p_{m+3}$.

The $P_m$-extended of a Star

As an illustration, the $P_m$-extended of $S_n$ is exhibited in Figure 1.
In the following, the vertex \( x \) marked by \( \Delta \) (respectively, \( \times \)) means \( x \in I \) (respectively, \( x \notin I \)) for a maximal independent set \( I \).

**Theorem 3.** For positive integers \( n \geq 1, m \geq 4 \),

\[
S_n(m) = p_{m-2} \cdot (p_{m-1})^{n-1} + p_{m-4} \cdot (p_m)^{n-1} - p_{m-4} \cdot (p_{m-3})^{n-1}.
\]

**Proof.** Let the \( P_m \)-extended of \( S_n \) be the graph of Figure 1. To obtain \( S_n(m) \), we divide all the maximal independent sets \( I \) in \( S_n(m) \) into two cases according to the vertex \((1,1)\) whether contains in \( I \):

Case 1. \((1,1) \in I\). Then \((1,i) \notin I, i = 2,3, \ldots n \) and \((2,1) \notin I \).

The number of maximal independent sets of \( S_n(m) \) is \( p_{m-2} p_{m-1} p_m \cdots p_{m-1} \) \( n-1 \) terms.

Case 2. \((1,1) \notin I\). We distinguish two subcases to consider.

The number of maximal independent sets of \( S_n(m) \) \( - (1,1) \) is \( p_{m-1} p_m p_m \cdots p_m \) \( n-1 \) terms.

The number of maximal independent sets of \( S_n(m) \) \( - (4,1) \) \( \cup \bigcup_{j=1}^{n} \{(i,j)\} \) is \( p_{m-4} p_{m-3} p_{m-3} \cdots p_{m-3} \) \( n-1 \) terms.

Thus we have that

\[
S_n(m) = n_{i+(1,1)}(S_n(m)) + n_{i-(1,1)}(S_n(m))
\]

\[
= p_{m-2} p_{m-1} p_m \cdots p_m + (p_{m-1} p_m p_m \cdots p_m - p_{m-4} p_{m-3} p_{m-3} \cdots p_{m-3})
\]

\[
= p_{m-2}(p_{m-1})^{n-1} + p_{m-1}(p_m)^{n-1} - p_{m-4}(p_{m-3})^{n-1}.
\]

This completes the proof.
**The $P_m$-extended of a Path**

As an illustration, the $P_m$-extended of $P_n$ is exhibited in Figure 2.

![Figure 2. $P_m$-extended of $P_n$.](image)

**Theorem 4.** $P_0(m) = 1$, $P_1(m) = P_m$, $P_2(m) = P_{2m}$, $m \geq 1$.

**Proof.** It follows the fact that $P_0(m) = P_0$, $P_1(m) = P_m$ and $P_2(m) = P_{2m}$.

**Theorem 5.** For positive integers $n \geq 3$ and $m \geq 3$,

$$P_n(m) = P_{m-1} \cdot P_{m-2} \cdot P_{m-2}(m) + P_{m-3} \cdot P_{m-1}(m) + P_{m-4} \cdot P_{m-2} \cdot (P_{m-4} - P_{m-3}) \cdot P_{m-3}(m).$$

**Proof.** Let the $P_m$-extended of $P_n$ be the graph of Figure 2. To obtain $P_n(m)$, we divide all the maximal independent sets $I$ in $P_n(m)$ into two cases according to the vertex $(1,1)$ whether contains in $I$:

**Case 1.** $(1,1) \in I$. Then $(1,2), (2,1) \notin I$.

**Case 2.** $(1,1) \notin I$. We distinguish three subcases to consider.

![Diagram](image)
If $(1, 2) \in I$, then $(1, 3), (2, 2) \notin I$. There are $P_{n-1}P_{n-2}P_{n-1}P_{n-3}(m)$ such sets.

If $(1, 2), (2, 1) \in I$, then $(1, 3), (2, 2), (3, 1) \notin I$. There are $P_{n-3}P_{n-2}P_{n-1}P_{n-3}(m)$ such sets.

By Case 2, we have that

$$m_i-1_{(1, 1)}(P_n(m)) = p_{n-3}P_{n-1}(m) + p_{n-1}p_{n-2}p_{n-1}P_{n-3}(m)$$

$$- p_{n-3}p_{n-2}p_{n-1}P_{n-3}(m)$$

$$= p_{n-3}P_{n-1}(m) + p_{n-1}p_{n-2}(p_{n-1} - p_{n-3})P_{n-3}(m).$$

Thus, by Lemma 1.2, we have that

$$P_n(m) = mi_{1(1, 1)}(P_n(m)) + mi_{-1(1, 1)}(P_n(m))$$

$$= p_{n-1}p_{n-2}P_{n-3} - 1) + p_{n-3}P_{n-1}(m) + p_{n-1}p_{n-2}(p_{n-1} - p_{n-3})P_{n-3}(m).$$

This completes the proof.

<table>
<thead>
<tr>
<th>$m$</th>
<th>$n$</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>1</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td>2</td>
<td>3</td>
<td>5</td>
<td>9</td>
<td>17</td>
<td>33</td>
<td>65</td>
<td>129</td>
<td>257</td>
<td>513</td>
</tr>
<tr>
<td>3</td>
<td>3</td>
<td>2</td>
<td>5</td>
<td>11</td>
<td>23</td>
<td>47</td>
<td>95</td>
<td>191</td>
<td>383</td>
<td>767</td>
<td>1535</td>
</tr>
<tr>
<td>4</td>
<td>4</td>
<td>3</td>
<td>9</td>
<td>25</td>
<td>69</td>
<td>193</td>
<td>549</td>
<td>1585</td>
<td>4629</td>
<td>13633</td>
<td>40389</td>
</tr>
<tr>
<td>5</td>
<td>5</td>
<td>4</td>
<td>16</td>
<td>62</td>
<td>238</td>
<td>914</td>
<td>3526</td>
<td>13682</td>
<td>53398</td>
<td>209474</td>
<td>825286</td>
</tr>
<tr>
<td>6</td>
<td>6</td>
<td>5</td>
<td>28</td>
<td>140</td>
<td>676</td>
<td>3236</td>
<td>15508</td>
<td>74660</td>
<td>361396</td>
<td>175896</td>
<td>8597908</td>
</tr>
<tr>
<td>7</td>
<td>7</td>
<td>7</td>
<td>49</td>
<td>327</td>
<td>2161</td>
<td>14343</td>
<td>96049</td>
<td>649287</td>
<td>4425841</td>
<td>13633</td>
<td>40389</td>
</tr>
<tr>
<td>8</td>
<td>8</td>
<td>9</td>
<td>86</td>
<td>764</td>
<td>6626</td>
<td>57164</td>
<td>494306</td>
<td>4296044</td>
<td>37549346</td>
<td>329544444</td>
<td>2912925026</td>
</tr>
<tr>
<td>9</td>
<td>9</td>
<td>12</td>
<td>151</td>
<td>1763</td>
<td>20155</td>
<td>230051</td>
<td>2640331</td>
<td>30531443</td>
<td>355654555</td>
<td>4169599811</td>
<td>49142154091</td>
</tr>
<tr>
<td>10</td>
<td>10</td>
<td>16</td>
<td>265</td>
<td>4123</td>
<td>62989</td>
<td>961051</td>
<td>14738365</td>
<td>227612203</td>
<td>3539594029</td>
<td>55380618811</td>
<td>870869975965</td>
</tr>
</tbody>
</table>

This completes the proof.
References


