Solving Mixed Quasi-variational Inequality for Modified Total Generalized Variation for Image Denoising

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Abstract. Image denoising has always been a hot issue in the field of image processing. In order to better reduce the image noise, a modified total generalized variation model is presented. The theory of mixed quasi-variational inequality is introduced to numerically solve the modified model. Compared with several variational models, experiments demonstrate that the proposed method has better performance in edge-preserving while filtering the noise.

Introduction

As we all known, various noises are mainly generated during the process of acquiring and transmitting digital image, which represents the quality of image is degraded. Image denoising is a process of acquiring the original clean image from noisy image, where the core course is how to retain more details while removing noise. Image denoising methods have been applied in various areas such as image restoration and 3 dimensional reconstruction, pattern recognition, and so on. In theory, image noise is defined as random error that cannot be predicted, however, it can be recognized by probabilistic statistical methods.

In mathematical sense, image denoising is a highly morbid inverse problem, which has no analytical solution or the solution is not unique or unstable[1-2]. Traditional image denoising methods, such as median filtering and Gauss filtering, mainly filter the high frequency components of image. Meanwhile, some details of the image are removed inevitably, which result in blurs. Different from the traditional filtering methods, variational methods, via minimizing the energy functional with image information, have been extremely successful in a variety of image denoising problems. The success makes variational methods be one of the most active research hotspots in image processing [3]. In order to remedy the deficiency mentioned above, Bradies, et al. introduced a new regularization approach called total generalized variation (TGV, in short) with high order [4], which eliminated the influence of staircase effect and improved the denoising performance. Recently, the theory of variational inequality [5-6] has expanded an explosive growth of algorithmic research, such as mixed variational inequality, quasi-variational inequality, and mixed quasi-variational inequality.

However, few works applied the theory of variational inequality to handle the problem of image denoising. Lenzen [7-8] proposed a class of quasi-variational inequalities for image restoration, where the adaptivity was described by solution-dependent constraint sets. Due to discontinuous solutions existing in bounded variation (BV, in short) space, the edge information of image can be retained at the same time of removing noise. In order to study the image deblurring problem, Yang [2] introduced a generalized TV regularization which was solved by mixed quasi-variational inequality. Inspired by the above studies [2,7-8], our discussion focuses on how to solve mixed quasi-variational inequality for modified TGV regularization approach.

We proposed a modified second-order TGV model. After that, mixed quasi-variational inequality is used to solve the modified TGV model. Finally, experimental results show that the modified TGV model proposed stands out from other regularization approaches.
The Modified TGV Model

Assume that the noise-free image $u$ is a real function defined on a bounded and piecewise smooth open subset (image domain) $\Omega \subseteq \mathbb{R}^2$, that is, $u: \Omega \to \mathbb{R}$. Denote by $f$ the noisy image, which can be obtained from image $u$ with an addition Gaussian noise

$$f = u + \delta,$$  \hspace{1cm} (1)

where $\delta$ represents Gaussian white noise with zero mean and standard deviation $\sigma$. Thus, given the noisy image $f$, we are interested in recovering $u$, which is well known to be an ill-posed problem, in general. Considering a discrete formulation of the denosing problem, assume that $u, f \in \mathbb{R}^n$ are the function values at the $n$ nodes of an equidistant 2D grid of size $M \times N$ on $\Omega$.

**Definition 1** Let $\Omega \subseteq \mathbb{R}^2$ denote a domain and $u: \Omega \to \mathbb{R}$ be a function defined on the domain. The total generalized variation of the function is given by

$$TGV^m(u) := \left\{ \frac{1}{2} \int_{\Omega} \text{div}^m \varphi \, dx : \varphi \in C^m_c\left( \Omega, \text{Sym}^m\left( \mathbb{R}^2 \right) \right), \| \text{div}^m \varphi \|_\infty \leq \lambda_m \right\},$$  \hspace{1cm} (2)

where $m \geq 1$; $\text{Sym}^m\left( \mathbb{R}^2 \right)$ represents the space of the $m$ order symmetric tensor with arguments in $\mathbb{R}^2$; $C^m_c\left( \Omega, \text{Sym}^m\left( \mathbb{R}^2 \right) \right)$ denotes the space symmetric tensor field supported tightly; and $\lambda = (\lambda_1, ..., \lambda_m) > 0$ are fixed positive weight parameters.

Compared with other regularization forms, TGV can be adapted to adjust the values according to the local conditions of image. In the smooth region of image, the second order gradient $\nabla^2 u$ becomes very small, and $TGV^2$ approximates to select $\vartheta = \nabla u$; in the edge region, the energy of the second-order gradient $\nabla^2 u$ will be greater than first-order, because $TGV^2$ approximates to select $\vartheta = 0$. Thus, TGV regularization form can fully display the strengths of high order variation and effectively eliminate the staircase effect for the task of image denoising, meanwhile better preserving edges and details[1,4-5,9]. Before the modified model is put forward, we discrete the TGV. Literature [5] noted that TGV can be interpreted as the dual semi-norm, where the set

$$\Pi^m_\lambda(\Omega) = \left\{ \text{div}^m \varphi : \varphi \in C^m_c\left( \Omega, \text{Sym}^m\left( \mathbb{R}^2 \right) \right), \| \text{div}^m \varphi \|_\infty \leq \lambda_m \right\}$$  \hspace{1cm} (3)

is a “predual unit ball”.

From Eq. (3), it can be seen that $\Pi^m_\lambda$ is convex. Recall that the energy functional of image denoising with second order TGV is given by the non-smooth minimization problem

$$\min_{\varphi \in H^1(\Omega)} \frac{1}{2} \| Ku - f \|_2^2 + TGV^2(\varphi).$$  \hspace{1cm} (4)

The solutions of formula (4) can be solved by Fenchel predual problem, which can be rewritten as a projection problem[1,4-5,9],

$$\min_{\varphi \in H^1(\Omega)} \frac{1}{2} \| f - \text{div}^2 \varphi \|_2^2 + I_{\|\text{div}\|_\infty \leq \lambda_2}\left( \varphi \right) + I_{\|\text{level}\|_\infty \leq \lambda_1}\left( \varphi \right).$$  \hspace{1cm} (5)

Solutions $u^*$ and $\varphi^*$ of formula (4) and formula (5), respectively, satisfy

$$u^* = K^{-1}\left( f - \text{div}^2 \varphi^* \right).$$  \hspace{1cm} (6)

We utilize a two-dimensional regular Cartesian grid with the size of $M \times N$ to represent the input image. The discrete analogue of formula (5) is given by

$$\min_{\varphi \in V} \left\{ \frac{1}{2} \| f - \text{div}^2 \varphi \|_2^2 \right\}, \quad T = \left\{ \varphi \in V : \| \varphi \|_\infty \leq \lambda_2, \| \text{div} \varphi \|_\infty \leq \lambda_1 \right\}.$$  \hspace{1cm} (7)
According to the duality principle put forward by Legendre-Fenchel, a dual formulation of formula (4) can be equivalently shown as

\[ \min_{u \in U} \left\{ \frac{1}{2} \| Ku - f \|_2^2 + \sup_{\varphi \in \Theta} \langle \text{div}^2 \varphi, u \rangle \right\}. \]  

(8)

Analogous to [2,7], we rewrite TGV in terms of constraint sets:

\[ TGV^2(u) = \sup_{c \in \text{div}^2 G} \left\{ \langle u, \varphi \rangle_{L^2}, \varphi \in c \right\}, \]  

(9)

where \( \text{div}^2 \) is applied elementwise on \( G \). Reference to [7-8], the dual problem (4) in the discrete formulation then becomes

\[ \min_{\varphi \in G} F(\varphi), \quad F(\varphi) = \frac{1}{2} \| L_0 \varphi - f \|_2^2, \]  

(10)

where \( \varphi = (\varphi_1, \ldots, \varphi_n)^T \in \mathbb{R}^n, \varphi_i \in \mathbb{R}^n \); \( L : \mathbb{R}^n \to \mathbb{R}^n \) is discretization of an divergence operator \( \text{div}^2 \). We assign a weight to put forward the modified TGV model, which is given as:

\[ \text{MTGV}^2(u) = TGV^2(u) - \beta \| KL \varphi \|_2 \]  

(11)

where \( \beta \) denote the regularization parameters. Hence, the optimization problem can be written as

\[ \arg \min_{u \in U} E(u), E(u) = \frac{1}{2} \| Ku - f \|_2^2 + \text{MTGV}^2(u) \]  

\[ = \frac{1}{2} \| Ku - f \|_2^2 + TGV^2(u) - \beta \| KL \varphi \|_2, \]  

(12)

where the first item is fidelity term, the second is the second order TGV regularization term, and the third item is supplementary term. Referring to literature[2,7], the Energy function can be rewritten as

\[ \arg \min_{u \in U} E(u), E(u) = \frac{1}{2} \| \tilde{K} u - \tilde{f} \|_2^2 + \sup_{\varphi \in G} \left\{ (L \varphi)^\top u - \beta \| KL \varphi \|_2 \right\}. \]  

(13)

On the basis of the optimality condition for \( u \), we can get

\[ \tilde{K}^\top (\tilde{K} u - \tilde{f}) + L \varphi = 0, \]  

(14)

which can be deduced as

\[ u = (\tilde{K}^{-1} \tilde{K}^\top) (\tilde{K}^\top \tilde{f} - L \varphi) = \tilde{K}^{-1} (\tilde{f} - \tilde{K}^{-1} L \varphi). \]  

(15)

Define \( A = \tilde{K}^{-1} L \), \( B = KL \). Combining (15) with (13), we obtain
\[ E'(\varphi) = \frac{1}{2} \left\| \mathcal{K}u - \tilde{f} \right\|_2^2 + (L\varphi)^\top u - \beta \left\| \mathcal{K}\varphi \right\|_2 \]
\[ = \frac{1}{2} \left\| \mathcal{K}^{-1}L\varphi \right\|_2^2 + (\mathcal{K}^{-1}\tilde{f})^\top L\varphi - (\mathcal{K}^{-1}\mathcal{K}^{-1}L\varphi)^\top L\varphi - \beta \left\| B\varphi \right\|_2 \]
\[ = \frac{1}{2} \left\| L\varphi \right\|_2^2 + \tilde{f}^\top \mathcal{K}^{-1}L\varphi - (\mathcal{K}^{-1}\mathcal{K}^{-1}L\varphi)^\top L\varphi - \beta \left\| B\varphi \right\|_2 \]
\[ = \frac{1}{2} \left\| A\varphi \right\|_2^2 + \tilde{f}^\top A\varphi - (A\varphi)^\top A\varphi - \beta \left\| B\varphi \right\|_2 \]
\[ = -\frac{1}{2} \left\| A\varphi \right\|_2^2 - 2\tilde{f}^\top A\varphi - \beta \left\| B\varphi \right\|_2 \]
\[ = -\frac{1}{2} \left\| (A\varphi - \tilde{f})^\top A\varphi - \tilde{f}^\top - \left\| \tilde{f} \right\|_2^2 \right\| - \beta \left\| B\varphi \right\|_2 \]
\[ = -\frac{1}{2} \left\| A\varphi - \tilde{f} \right\|_2^2 + \frac{1}{2} \left\| \tilde{f} \right\|_2^2 - \beta \left\| B\varphi \right\|_2 \] (16)

From Eq. (15), we can see that \( E'(\varphi) \) is a function of \( \varphi \). Then, when maximizing \( E' \) over \( G \), the second item \( \frac{1}{2} \left\| \tilde{f} \right\|_2^2 \) is a constant term without changing the optimum, which can be neglected. The dual problem of formula (13) can be derived as

\[ \operatorname{argmin}_{\varphi \in G} J(\varphi) = \frac{1}{2} \left\| A\varphi - \tilde{f} \right\|_2^2 + \beta \left\| B\varphi \right\|_2, \beta \in [0,1]. \] (17)

Lenzen put forward the solution-driven adaptivity\[7-8\], where the constraint set \( G \) depends on the unknown solution \( \varphi \). According to the literature\[2,7\], we can see that the adaptivity is determined by the noise-free image. To study the modified TGV method, by introducing a dependency of \( G \) on the dual variable, formula (17) can be generalized: find a fixed point \( \varphi^* \) of the mapping,

\[ \varphi_0 \mapsto \varphi := \operatorname{argmin}_{\varphi \in G(\varphi_0)} J(\varphi) = F(\varphi) + \beta \phi(\varphi), \quad F(\varphi) = \frac{1}{2} \left\| A\varphi - \tilde{f} \right\|_2^2, \quad \phi(\varphi) = \left\| B\varphi \right\|_2. \] (18)

After having found a fixed point \( \varphi^* \), the corresponding constraint set is \( G(\varphi^*) \); i.e., the adaptivity becomes solution-driven, and we can retrieve the primal solution \( u^* \) as the solution of (17) with fixed \( \varphi_0 = \varphi^* \). Solving the problem (18) is equivalent to considering the following mixed quasi-variational inequality problem (MQVIP, in short) \[2,5-6\] of find \( \varphi^* \in G(\varphi^*) \) such that

\[ \langle \nabla F(\varphi^*), \varphi - \varphi^* \rangle + \beta \phi(\varphi) - \beta \phi(\varphi^*) \geq 0, \quad \forall \varphi \in G(\varphi^*), \] (19)

where \( \nabla F(\varphi) = A'(A\varphi - \tilde{f}) \).

**Experimental Results**

In this section, the denoising performance of the proposed method will be furtherly evaluated. In order to compare the effectiveness with various denoising methods, the standard test images polluted by adding Gaussian noise with zero mean (with the size of 256×256) were chosen as the experimental images. For noise removal, the proposed method is compared with other models, such as the TV\[10\], TGV \[11-12\]. In this paper, the peak signal-to-noise ratio (PSNR) value and mean structure similarity (MSSIM) are taken as the evaluation indicators to estimate the denoising performance.

The simulations are implemented in Matlab R2014a on a laptop equipped with 2.50 GHz CPU and 4G RAM memory. In order to show the denoising effect better, we take the three test images with Gaussian noise standard deviation 20 as examples, shown in Fig.1. It shows the denoising results of TV, TGV and proposed model. Zoom-in regions of the filtered images are also shown in Fig.1. Table 1 shows the comparison results of PSNR value and MSSIM value of the filtered images with different methods. From Table 1, we can see that PSNR and MSSIM values with the proposed method are slightly larger than the values with TV and TGV methods. Consequently, we believe that the proposed method can perform better than other regularization ones in removing Gaussian noise.
Table 1. Comparison results of image denosing.

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Figure 1. Denosing results of the noisy Camera man image. (a) noisy image (PSNR=24.48, MSSIM=0.72); (b) zoom-in of (a); (c) TV (PSNR=29.41, MSSIM=0.87); (d) zoom-in of (c); (e) TGV (PSNR=29.56, MSSIM=0.85); (f) zoom-in of (e); (g) proposed method (PSNR=30.65, MSSIM=0.89); (h) zoom-in of (g).

From Fig.1 and Table 1, we can observe that the three methods above all can be used to effectively remove the Gaussian noise while preserving more details. For the noisy camera man image, the PSNR is 24.48 and MSSIM is 0.72. For the image filtered by TV method, the PSNR is 29.41 and MSSIM is 0.87. For the image filtered by TGV method, the PSNR is 29.56 and MSSIM is 0.85, while the PSNR is 30.65 and MSSIM is 0.89 with the proposed method. However, the images filtered by TV model appear staircase effect. The images with TGV method have better qualities than the performances by TV. Obviously, the proposed method has better effect in staircase reduction than TV method, and in edge-preserving than TGV method. It means that the proposed method suppresses the Gaussian noise effectively and performs significantly better than other methods in the ways of preserving more details and edges of image.

Conclusion

In this paper, we proposed a modified TGV method for image denoising. Then we discussed the mixed quasi-variational inequality to solve the solution of the modified TGV model. Experiment results showed that the proposed denoising method stands out from other regularization approaches while preserving more details and edges.

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References


