Single-pulse Chaotic Dynamics in Suspended Cables with Parametric and External Excitations

Yu-gao HUANGFU* and Guo-ying YANG

College of Mathematics and Information Science, Henan Polytechnic University,
Jiaozuo 454000, PR China

*Corresponding author

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Abstract. The chief aim of the present work is to investigate the single-pulse chaotic dynamics for suspended cables with principal parametric and primary resonant excitation. Using higher-dimensional Melnikov theory, explicit conditions for the existence of single-pulse Shilnikov-type homoclinic orbits are obtained, which imply that chaotic motions in the sense of Smale horseshoes may occur for this system. Finally, numerical results are presented to verify these analytical predictions.

Introduction

The suspended cables are widely used in engineering, such as communications, transportation, cable-stayed bridge etc. In the past two decades, research works on it have received considerable attention [1–5]. Due to the development of the theories of nonlinear dynamics and chaos, prediction of the global bifurcations and the Shilnikov-type chaotic dynamics becomes possible for the suspended cable systems.

In [6], four dimensional perturbed Hamiltonian systems were divided into three basic types, and Melnikov method was used to analyze the global bifurcations and chaotic dynamics of these three types systems. Based on the study given by Wiggins[6], a new global perturbation method which could be utilized to detected the existence of single-pulse homoclinic and heteroclinic orbits for some four dimensional systems was presented by Kovacic and Wiggins [7]. This global perturbation technique is a combination of higher-dimensional Melnikov theory, geometrical singular perturbation theory and the theory of invariant manifolds. It is considered that this technique is fairly effective when the Shilnikov-type single-pulse homoclinic and heteroclinic orbits are studied in four-dimensional nonlinear systems. Several researchers applied this technique to many engineering problems, see [3, 4, 8–18] for more details.

In [4,5], using the method of multiple scales, a parametrically and externally excited suspended cables system was transformed to the averaged equations, and some local bifurcation behaviors were presented in [4]. In [5], Chen et.al had studied the multipulse homoclinic orbits in the case $f_1 = f_2, k_2 = k_5, k_3 = k_6, c_1 = c_5$ of this suspended cables system, but we think it is necessary to consider further. Hence, based on their researches, in this paper, some cases of this system are considered by the method of normal form, and the studies of this paper focus on the single pulse homoclinic orbits and chaos for this system. This paper is organized as follows: Govern equations of the suspended cables, averaged equations and normal form of the averaged equations are presented in Section 2. In Section 3, some dynamical properties of the unperturbed system on heteroclinic bifurcations are obtained. Analysis about the perturbed system is presented in Section 4, and existence of the single-pulse Shilnikov homoclinic orbits is given in Section 5. Some numerical simulations are given in Section 6 to verify our conclusions in Section 5. The work ends in Section 7 with a short summary.
Normal Form

A shallow suspended elastic cable hanging at fixed supports and caused by axial excitation and horizontal load is considered. The non-dimensional non-planar equations of motion can be written as (see [4, 5] for more details)

\[
\begin{align*}
\ddot{u}_1 + 2\gamma \dot{u}_1 - u_1' &= (\alpha u_1^2 + \beta u_1^3) \int_0^L [F_1 \cos \Omega t + b_1' \dot{u}_1' + \frac{1}{2}(\alpha u_1^2 + \alpha u_1^3)] dx, \\
\ddot{u}_2 + 2\gamma \dot{u}_2 - u_2' &= \alpha u_2 \int_0^L [F_1 \cos \Omega t + b_1' \dot{u}_1' + \frac{1}{2}(\alpha u_1^2 + \alpha u_1^3)] dx + f(x) \cos \Omega t.
\end{align*}
\]

(1)

Where \( \psi(x) = 4x(1-x) \) and the boundary conditions are

\[
u_1 = u_2 = 0, \quad \text{at } x = 0 \text{ and } x = 1.
\]

(2)

The overdot and prime indicate the derivatives with respect to \( t \) and \( x \).

Let

\[
u_1(x,t) = \phi_n(x) x(t), \quad \nu_2(x,t) = \phi_n(x) y(t).
\]

(3)

where the \( \phi_n \) and \( \phi_n^* \) are the mode shapes of the linearized problem, \( x \) and \( y \) are generalized coordinates. Using the Galerkin procedure, the system (1) can be discretized to a two-degree-of-freedom nonlinear equation

\[
\begin{align*}
\ddot{x} + 2\dot{c}_x \dot{x} + \left(\omega_x^2 x + \varepsilon \gamma_2 x^2 + \varepsilon^2 \gamma_2 y^2 + \varepsilon^2 \gamma_2 n^2 y^2\right) &= 0, \\
\ddot{y} + 2\dot{c}_y \dot{y} + \left(\omega_y^2 y + \varepsilon \gamma_2 y^2 + \varepsilon^2 \gamma_2 x^2 + \varepsilon^2 \gamma_2 n^2 x^2\right) &= 0,
\end{align*}
\]

(4)

where the \( \dot{x} \) and \( \dot{y} \) refer to the in-plane and out-of-plane displacements, respectively. \( c_x \) and \( c_y \) are viscous damping coefficients, \( 0 < \varepsilon << 1 \) is a small parameter. \( \omega_x \) and \( \omega_y \) represent the natural frequencies associated with the antisymmetric or symmetric in-plane and out-of-plane modes.

In the system (4), we assume \( f_2 = \varepsilon f_1 \). The primary resonance and principal parametric excitations in the presence of one-to-one internal resonance is considered, and the detuning parameters \( \sigma_1 \) and \( \sigma_2 \) are introduced as \( \omega_x = \Omega - \varepsilon^2 \sigma_1 \), \( \omega_y = \Omega - \varepsilon^2 \sigma_2 \) and \( \Omega_1 = 2\Omega \). The second order approximate solutions of (4) can be obtained by using the method of multiple scales, after some algebraic calculations, the averaged equations in the Cartesian forms can be written as

\[
\begin{align*}
x_1' &= -2c_1 x_1 - (\sigma_1 - f_1 - f_2) x_1 - 2\varepsilon \gamma_2 \Omega_1 x_1 - (\gamma_2 x_1^2 + \gamma_2 y_1^2 + \gamma_2 n^2 y_1^2 + \gamma_2 n^2 x_1^2) x_1, \\
x_2' &= (\sigma_1 + f_1 + f_2) x_1 - 2\varepsilon \gamma_2 \Omega_1 x_1 - (\gamma_2 x_1^2 + \gamma_2 y_1^2 + \gamma_2 n^2 y_1^2 + \gamma_2 n^2 x_1^2) x_1, \\
x_3' &= -2c_1 x_3 - (\sigma_1 - f_1 - f_2) x_3 - 2\varepsilon \gamma_2 \Omega_1 x_3 - (\gamma_2 x_3^2 + \gamma_2 y_3^2 + \gamma_2 n^2 y_3^2 + \gamma_2 n^2 x_3^2) x_3, \\
x_4' &= (\sigma_1 + f_1 + f_2) x_3 - 2\varepsilon \gamma_2 \Omega_1 x_3 - (\gamma_2 x_3^2 + \gamma_2 y_3^2 + \gamma_2 n^2 y_3^2 + \gamma_2 n^2 x_3^2) x_3.
\end{align*}
\]

(5)

where coefficients \( k_1, k_2, k_3, k_4, k_5, k_6 \) present in (5) can be found in paper [4, 5].

To analyze the global bifurcations and chaotic dynamics for the suspended cables, one need to reduce averaged equation (5) to a simpler normal form. It is seen that there are \( Z_2 \otimes Z_2 \) and \( D_4 \) symmetries in averaged equations (5) without the parameters. Therefore, these symmetries are held in normal form too.

Take the exciting amplitude \( f \) as a perturbation parameter, so \( f \) can be considered as an unfolding parameter when the global bifurcations are investigated. When the perturbation parameter not be considered, equation (5) becomes

\[
\begin{align*}
x_1' &= -2c_1 x_1 - (\sigma_1 - f_1 - f_2) x_1 - 2\varepsilon \gamma_2 \Omega_1 x_1 - (\gamma_2 x_1^2 + \gamma_2 y_1^2 + \gamma_2 n^2 y_1^2 + \gamma_2 n^2 x_1^2) x_1, \\
x_2' &= (\sigma_1 + f_1 + f_2) x_1 - 2\varepsilon \gamma_2 \Omega_1 x_1 - (\gamma_2 x_1^2 + \gamma_2 y_1^2 + \gamma_2 n^2 y_1^2 + \gamma_2 n^2 x_1^2) x_1, \\
x_3' &= -2c_1 x_3 - (\sigma_1 - f_1 - f_2) x_3 - 2\varepsilon \gamma_2 \Omega_1 x_3 - (\gamma_2 x_3^2 + \gamma_2 y_3^2 + \gamma_2 n^2 y_3^2 + \gamma_2 n^2 x_3^2) x_3, \\
x_4' &= (\sigma_1 + f_1 + f_2) x_3 - 2\varepsilon \gamma_2 \Omega_1 x_3 - (\gamma_2 x_3^2 + \gamma_2 y_3^2 + \gamma_2 n^2 y_3^2 + \gamma_2 n^2 x_3^2) x_3.
\end{align*}
\]

(6)
It is obviously known that equation (6) has a trivial zero solution \((x_1, x_2, x_3, x_4) = (0, 0, 0, 0)\), at which the Jacobi matrix can be written as

\[
J = \begin{bmatrix}
-c_x & -(c_1 - f_1) & 0 & 0 \\

c_1 + f_1 & -c_x & 0 & 0 \\
0 & 0 & -c_y & -c_2 \\
0 & 0 & -c_y & c_2
\end{bmatrix}.
\]  

(7)

The characteristic equation corresponding to the trivial zero solution is of the form

\[
(\lambda^2 + 2c_2 \lambda + c_2^2 + c_1^2 - f_1^2)(\lambda^2 + 2c_2 \lambda + c_2^2 + c_2^2) = 0.
\]  

(8)

Let

\[
\Delta_1 = c_2^2 + c_1^2 - f_1^2, \quad \Delta_2 = c_2^2 + c_2^2.
\]  

(9)

When \(c_y = c_x = 0\) and \(\Delta_1 = \sigma_1^2 - f_1^2 = 0\) are satisfied simultaneously, system (6) has a double zero and a pair of pure imaginary eigenvalues

\[
\lambda_{1,2} = 0, \lambda_{3,4} = \pm i \omega.
\]  

(10)

Where \(\omega^2 = \sigma_2^2\).

Let \(\sigma_i = -f_i + \sigma\) and \(f_1 = \frac{1}{2}\). Taking \(\sigma, c_y, c_x, f\) as the perturbation parameters, then, the averaged equation (6) without the perturbation parameters is changed to the following form

\[
x_1 = x_2 - 2k_4 y_1 x_2 x_3 - (k_1 x_1^2 + k_1 x_3^2) - (k_2 - k_3) x_4^2 + (k_2 + k_3) y_2 y_3 x_4,
\]

\[
x_2 = 2k_2 x_3 y_1 + (k_1 x_1^2 + k_1 x_3^2) - (k_2 - k_3) y_2^2 x_4 + (k_2 + k_3) y_2 y_3 x_4,
\]

\[
x_3 = -x_4 x_1 - 2k_4 y_1 x_1 x_3 + (k_1 x_1^2 + k_1 x_3^2) - (k_2 - k_3) x_4^2 + (k_2 + k_3) y_2 y_3 x_4,
\]

\[
x_4 = x_2 x_3 + 2k_2 y_1 y_2 x_4 + (k_1 x_1^2 + k_1 x_3^2) + (k_2 - k_3) y_2^2 x_4 + (k_2 + k_3) y_2 y_3 x_4 + f.
\]  

(11)

Executing the Maple program given by Zhang et al. [19], the three-order normal form of system (11) obtained as

\[
y_1' = y_2,
\]

\[
y_2' = k_1 y_1^2 + k_2 y_2 y_1 + k_3 y_3 y_4,
\]

\[
y_3' = -k_2 y_1^2 - k_3 y_2^2 y_4 - k_4 y_2 y_4,
\]

\[
y_4' = k_2 y_2 y_4 + k_3 y_3 y_4 + k_4 y_2 y_3 y_4.
\]  

(12)

The nonlinear transformation used in the above computing procedure is of the form

\[
x_1 = \frac{x_1}{c_y} - \frac{1}{2}k_4 y_1 x_2 x_3 + k_1 x_1^2 x_3 - h_2 y_2 y_4 - h_3 y_2 y_3 x_4 + h_4 (1 - 2c_2 y_1 x_2 x_3 + 2c_2 y_2 y_3 x_4),
\]

\[
x_2 = \frac{x_2}{c_y} + \frac{1}{2}k_4 y_1 x_1 x_3 + k_1 x_1^2 x_3 - h_2 y_2 y_4 - h_3 y_2 y_3 x_4 + h_4 (1 - 2c_2 y_1 x_2 x_3 + 2c_2 y_2 y_3 x_4),
\]

\[
x_3 = \frac{x_3}{c_y} - \frac{1}{2}k_4 y_1 x_1 x_3 + k_1 x_1^2 x_3 - h_2 y_2 y_4 - h_3 y_2 y_3 x_4 + h_4 (1 - 2c_2 y_1 x_2 x_3 + 2c_2 y_2 y_3 x_4),
\]

\[
x_4 = \frac{x_4}{c_y} + \frac{1}{2}k_4 y_1 x_1 x_3 + k_1 x_1^2 x_3 - h_2 y_2 y_4 - h_3 y_2 y_3 x_4 + h_4 (1 - 2c_2 y_1 x_2 x_3 + 2c_2 y_2 y_3 x_4),
\]  

(13)

The normal form with perturbation parameters can be written as

\[
y_1' = c_y y_1 + (1 - \sigma) y_2,
\]

\[
y_2' = \sigma y_2 - c_y y_1 + k_1 y_1^2 + k_2 y_2 y_4 + k_3 y_3 y_4,
\]

\[
y_3' = -c_y y_1 - \sigma y_2 - k_2 y_2^2 - k_3 y_2 y_4 - k_4 y_2 y_4,
\]

\[
y_4' = k_1 y_1^2 + k_2 y_2 y_4 + k_3 y_3 y_4 + k_4 y_2 y_3 y_4 + f.
\]  

(14)

Further, one let

\[
y_3 = l \cos \varphi, \quad y_4 = l \sin \varphi.
\]  

(15)

Substituting equation (15) into (14) yields
In order to obtain the unfolding of equation (16), a linear transformation is introduced

\[
\begin{bmatrix}
y_1 \\
y_2
\end{bmatrix} = \begin{bmatrix}
\sqrt{\frac{k_2}{k_1}} & 0 \\
0 & \sqrt{\frac{c_2}{c_1}}
\end{bmatrix} \begin{bmatrix}
v_1 \\
v_2
\end{bmatrix}
\]

Substituting equation (17) and (18) into (16) and cancelling the nonlinear terms which include the parameter \( \sigma \) yield the unfolding as

\[
v'_1 = u_2, \\
v'_2 = \mu_1 u_1 - \mu_2 u_2 + \eta u_1^2 + k_2 u_1 I^2, \\
I' = -c_0 I + f \sin \varphi, \\
I'' = -c_0 I + k_3 I^2 + k_4 I^3 + f \cos \varphi.
\]

where \( x_2, x_3, x_4 \) and

In order to analysis the small perturbation of the system, one can re-scale system’s parameters as

\[
\mu_2 = \varepsilon \mu_2, \quad c_0 = \varepsilon c_0, \quad f = \varepsilon f.
\]

Then, the unfolding (19) can be rewritten as the Hamiltonian form with the perturbation

\[
v'_1 = \frac{\partial H}{\partial \varphi} + \varepsilon g^{n1} = u_2, \\
v'_2 = \frac{\partial H}{\partial u_1} + \varepsilon g^{n2} = \mu_1 u_1 - \mu_2 u_2 + \eta u_1^2 + k_2 u_1 I^2 - \varepsilon \mu_2 u_2, \\
I' = \frac{\partial H}{\partial I} + \varepsilon g' = -c_0 I + \varepsilon f \sin \varphi, \\
I'' = -\frac{\partial H}{\partial I} + \varepsilon g'' = -c_0 I + k_3 I^2 + \varepsilon f \cos \varphi.
\]

where the Hamiltonian function is

\[
H(u_1, u_2, I, \varphi) = \frac{1}{2} u_1^2 + \frac{1}{2} u_2^2 - \frac{1}{4} \mu_1 I^2 - \frac{1}{4} \mu_2 I^2 - \frac{1}{4} k_2 I^4.
\]

And \( g^{n1} = 0, g^{n2} = -\mu_2 u_2, g' = -c_0 I + f \sin \varphi, g'' = f \cos \varphi. \)

**Dynamics of the Unperturbed System**

When \( \varepsilon = 0 \), system (21) is an uncoupled two degree of freedom nonlinear system. The variable \( I \) appears in components \( (u_1, u_2) \) of system (21) can be regarded as a parameter because \( I' = 0 \). So, the first two decoupled equations with perturbation term are considered firstly

\[
v'_1 = \frac{\partial H}{\partial u_1} + \varepsilon g^{n1} = u_2, \\
v'_2 = \frac{\partial H}{\partial u_1} + \varepsilon g^{n2} = \mu_1 u_1 + \eta u_1^2 + k_2 u_1 I^2 - \varepsilon \mu_2 u_2.
\]

When \( \eta > 0 \), in coordinate plane \( (u_1, u_2) \), system (23) can exhibit the heteroclinic bifurcations. From equation (23), it is easy to know that when \( \mu_1 + k_2 I^2 > 0 \), the trivial zero solution \( (u_1, u_2) = (0,0) \) is the only solution of system (23), which is a saddle point. On the curve defined by \( \mu_1 = -k_2 I^2 \), that is

\[
\sigma^2 = \sigma(1 - \sigma) + k_2 I^2.
\]
the trivial zero solution may bifurcate into three solutions through a pitchfork bifurcation represented by \( q_0 = (0,0) \), and \( q_\pm = (B,0), \) respectively, where

\[
B = \pm \sqrt{\frac{\sigma^2 - \eta (1 - \sigma)}{k_2 \eta}}.
\]

By the Jacobian matrix evaluated at the nonzero solutions \( q_\pm (I) \), it is found that the singular points \( q_\pm (I) \) are the saddle points.

It is observed that variables \( I \) and \( \phi \) may actually represent the amplitude and phase of nonlinear oscillations. Therefore, one may assume that variable \( I \geq 0 \) and equation (25) becomes

\[
I_1 = 0, \quad I_2 = \sqrt{\frac{\sigma^2 - \eta (1 - \sigma)}{k_2}},
\]

Figure 1. Phase portrait in the \( \mu_1 - \mu_2 \) plane for the unperturbed system (23) with \( I = 1, \eta = 1, k_2 = 1, \mu_1 = -2 \).

such that for all \( I \in [I_1, I_2] \) , system (23) has two hyperbolic saddle points \( q_\pm \), which are connected by a pair of heteroclinic orbits \( u^\pm_h(t,I) \), that is, \( u^+_h(t,I) \to q_+(I) \) as \( t \to \pm \infty \). In the Fig.1, each of heteroclinic orbits satisfies

\[
\frac{1}{2} \ddot{\alpha}^2 + \frac{1}{2} \mu_1 \dot{\alpha}^2 - \frac{1}{2} k_2 u^2_\alpha^2 - \frac{1}{4} \eta u^2_\alpha = 0
\]

Therefore, in full four dimensional phase space the set defined by

\[
M_0 = \{(\alpha, I, \varphi) | \alpha = q_\pm (I), I_1 \leq I \leq I_2, 0 \leq \varphi \leq 2\pi \}
\]

is a two dimensional invariant manifold. From [7], it is known that two dimensional invariant manifold \( M_0 \) is normally hyperbolic with three dimensional stable and unstable manifolds which are respectively expressed as \( W^s(M_0) \) and \( W^u(M_0) \). The existence of the heteroclinic orbit of system (23) to the saddle points \( q_\pm (B,0) \) indicates that the stable and unstable manifolds \( W^s(M_0) \) and \( W^u(M_0) \) intersect nontransversally along a three dimensional heteroclinic manifold denoted by \( \Gamma \), of form

\[
\Gamma = \{(\alpha, I, \varphi) | \alpha = u^0_h(t,I), I_1 \leq I \leq I_2, \varphi = \int^{T} D_t H(u^0_h(t,I),I) ds + \alpha_0 \}.
\]

Now, the dynamics of the unperturbed system of equation (21) restricted to the manifold \( M_0 \) are considered. The unperturbed system (21) restricted to the manifold \( M_0 \) yields

\[
I_\varphi' = 0,
\]

\[
I_\varphi = D_t H(q_\pm (I),I) = -\sigma_1 I + \eta_2 \dot{I} \pm (B) + k_1 I, I_1 \leq I \leq I_2.
\]

By the results given by Kovacic and Wiggins [7], we know, if the condition \( D_t H(q_\pm (I),I) \neq 0 \) is satisfied, \( I = \text{cons tan} t \) is called a period orbit, and if the condition \( D_t H(q_\pm (I),I) = 0 \) is satisfied,
I = \cos t \tan t is called a circle of the singular points. A value of \( I \in [I_1, I_2] \) at which 
\( D_I H(q_{\pm}(I), I) = 0 \) is referred to as a resonant \( I \) value and these singular points as the resonant 
singular points. A resonant value is denoted by \( I_r \) so that

\[ D_I H(q_{\pm}(I), I_r) = -\sigma q_{\pm} + \frac{k_2 I}{y^3} - \sigma(1 - \sigma) - b_2 q_{\pm}^2 + k_4 q_{\pm}^4 = 0. \]  

Then, one can obtain a resonant value

\[ I_r = \sqrt{\frac{k_2(\sigma^2 - \sigma(1 - \sigma)) - \sigma_2 \eta}{k_2^2 - k_4 \eta}}. \]  

Figure 2. The geometric structure of manifolds \( M_0, W^s(M_0) \) in the full-dimensional phase space, (a) heterocline 
manifold, (b) the parital invariant manifold \( M \).

The geometry structure of the stable and unstable manifolds of \( M_0 \) in full four dimensional phase 
space for the unperturbed system (21) is given in Fig.2. Because variable \( \theta \) may represent the 
phase of nonlinear oscillations, when \( I = I_r \), the phase shift \( \Delta \theta \) of nonlinear oscillations is defined 
as

\[ \Delta \theta = \varphi(+\infty) - \varphi(-\infty). \]  

The physical interpretation of the phase shift is the phase difference between the two end points 
of the heteroclinic orbit along the circle of singular points. In \( (u_1, u_2) \) subspace, there exist a pair of 
the heteroclinic orbits connecting the two saddles. Therefore, the homoclinic orbit in subspace 
\( (I, \varphi) \) is of a heteroclinic connecting in full four dimensional phase space \( (u_1, u_2, I, \varphi) \) the phase 
shift represents the difference of \( \varphi \) value as a trajectory leaves and returns to the basin of attraction 
of the manifold \( M_0 \). To obtain the condition for the existence of the Silnikov type single-pulse heteroclinic orbit, 
the phase shift in subsequent analysis will be used. Hence, the phase shift 
will be calculated for the heteroclinic orbit in the later.

The heteroclinic bifurcations of equation (23) is considered now. Let \( \alpha = -(\mu_1 + k_2 I^2) \), equation 
(23) can be rewritten as

\[
\begin{align*}
  u_1' &= u_2, \\
  u_2' &= -\alpha u_1 + \eta u_1^3 + \varepsilon \mu_2 u_2.
\end{align*}
\]  

Setting \( \varepsilon = 0 \) in equation (35), then the equation (35) is a Hamiltonian system with Hamiltonian 
function

\[ H(u_1, u_2) = \frac{1}{2} u_2^2 + \frac{1}{2} \alpha u_1^2 - \frac{1}{4} \mu_2 u_2^4. \]  

As \( \frac{\mu_2}{4 \eta} \), there exists a heteroclinic loop \( \Gamma^0 \), which consists of the two hyperbolic saddles 
\( q_{\pm} \) and a pair of heteroclinic orbits \( u_{\pm}(I) \). The equations of a pair of heteroclinic orbits are obtained 
as

\[ H = \frac{\alpha^2}{4 \eta}, \]
Substituting the first equation of (37) into the fourth equation of the unperturbed system of equation (21) yields

\[ \varphi' = -\sigma_2 + k_d I^2 + \frac{k_2\alpha}{\eta} \tanh\left(\frac{\sqrt{2}\alpha}{2} t\right). \]  

Integrating equation (38) yields

\[ \varphi(t) = \omega_I t - \frac{k_2\sqrt{2}\alpha}{\eta} \tanh\left(\frac{\sqrt{2}\alpha}{2} t\right) + \varphi_0, \] 

where

\[ \omega_I = -\sigma_2 + k_d I^2 + \frac{k_2\alpha}{\eta}. \]  

At \( I = I_r \), there is \( \omega_I \equiv 0 \). Hence, the phase shift is

\[ \Delta \varphi = \left[ -\frac{2k_2\sqrt{2}\alpha}{\eta} \right]_{t \to I_r} = -\frac{2k_2}{\eta} \sqrt{2}\alpha^2 - \sigma(1 - \sigma) - k_d I^2. \]  

**Perturbed System**

Based on the results [6, 7], it is known that the manifold \( M_0 \) (heteroclinic case) or \( M_0^h \) (homoclinic case) along with its stable and unstable manifolds are invariant under sufficiently small perturbations. The singular points \( Q_\pm \) may persist under small perturbations also. So, we have

\[ M_0 = M_r = \{(u, l, \varphi) | u = g_k(l), I_1 < I < I_2, 0 < \varphi < 2\pi\}. \]  

Consider the latter two equations of (21) yields

\[ I' = -c\varphi, \quad \varphi' = -\sigma_2 + k_d I^2 + \frac{f}{\tau} \cos \varphi. \]  

From the aforementioned analysis, it is known that there is a pair of pure imaginary eigenvalues in equations (43). So, resonance can occur in system (43). Following the analysis of Kovacic and Wiggins [7] we introduce the following transformations in order to study the dynamics on \( M_{r, \varepsilon} \) near \( I = I_r \)

\[ c \to c, \quad f \to \varepsilon f, \quad I = l + \varepsilon h, \quad \tau = \sqrt{\varepsilon} \tau. \]  

Substituting the above transformations into equation (43), we have:

\[ \dot{h} = -c_l I_r + f \sin \varphi - \sqrt{\varepsilon} c, h, \]

\[ \dot{\varphi} = -\frac{2\delta}{\eta} l h - \sqrt{\varepsilon} \dot{h}^2 + \sqrt{\varepsilon} \frac{f}{\tau} \cos \varphi. \]  

where \( \delta = k_2^2 - k_d \eta \), and the dot denotes the differentiation with respect to \( \tau \).

In the limit \( \varepsilon \to 0 \), system (45) reduce to

\[ \dot{h} = -c_l I_r + f \sin \varphi = \frac{\partial H^*}{\partial \varphi}, \]

\[ \dot{\varphi} = -\frac{2\delta}{\eta} l h = -\frac{\partial H^*}{\partial h}. \]  

(45)
which is a Hamilton system with Hamiltonian
\[ H'(h, \varphi) = -c_y I_r \varphi - f \cos \varphi + \frac{5}{11} I_r h^2. \] (46)

The singular points of equation (46) can be written as
\[ p_0 = (0, \varphi_0) = (0, \arcsin \frac{c_y I_r}{f}) \quad \text{and} \quad q_0 = (0, \varphi_0) = (0, \pi - \arcsin \frac{c_y I_r}{f}). \] (47)

When \( k_4 < 0 \), linear stability analysis indicates, that the singular point \( P_0 \) is a center and \( Q_0 \) is a saddle which is connected to itself by a homoclinic orbit. We can show that the leading order term of the trace of the linearization of the perturbed system (45) is less than zero inside the homoclinic orbit connecting \( Q_0 \). In this case, under small perturbations, \( Q_0 \) remains a hyperbolic fixed point \( Q_0 \) of same stability type, while \( P_0 \) becomes a hyperbolic sink \( P_0 \). The homoclinic orbit breaks and the unstable branch of the perturbed saddle point gets attracted to the hyperbolic sink. All the periodic orbits inside unperturbed homoclinic orbit are generally destroyed for \( \epsilon \neq 0 \).

At \( h = 0 \), the estimate of basin of attraction for \( \varphi_{\min} \) is obtained as
\[ c_y I_r \varphi_{\min} + f \cos \varphi_{\min} = c_y I_r \varphi_*, f \cos \varphi_*. \] (48)

Substituting \( \varphi_* \) in equation (48) into equation (49) yields
\[ \varphi_{\min} + f c_y I_r \cos \varphi_{\min} = \frac{\pi - \arcsin \frac{c_y I_r}{f} - \sqrt{f^2 - c_y^2 I_r^2}}{c_y I_r}. \] (49)

We are interested in the dynamics on \( M_\epsilon \) in \( O(\epsilon) \) neighborhood of \( I_r \). This neighborhood can be defined as an annulus \( A_\epsilon \) as
\[ A_\epsilon = \{ (u_1, u_2, I, \varphi) | u_1 = h, u_2 = 0, |I - L| < \sqrt{C}, \varphi \in T \}. \] (50)

Where \( C \) is a constant, which is chosen sufficient large so that the unperturbed homoclinic orbit is enclosed within the annulus.

**Existence of Single-pulse Orbits**

Based on the results obtained in [7], the higher-dimensional Melnikov function is given as follows
\[ M_\epsilon = \int_{-\infty}^{\infty} \left[ \frac{\partial H}{\partial \eta} \eta^n + \frac{\partial H}{\partial \eta'} \eta' + \frac{\partial H}{\partial \varphi'} \varphi' \right] dt \]
\[ = \int_{-\infty}^{\infty} \left[ c_y I_r \varphi_1 + (c_y \varphi_1 - b_2 \varphi_1^3) + c_y I_r \varphi_1 + f \sin \varphi_1 \right] dt. \] (51)

Where \( u_1, u_2 \) and \( \varphi \) are respectively given in (37) and (39).

By the aforementioned analysis, the first and second integrands in (52) can be calculated as follows
\[ \int_{-\infty}^{\infty} -j \sqrt{2} \eta^3 \eta' \eta' dt = -\frac{2\sqrt{2} \eta^3 \eta' \eta'}{3\eta}. \] (52)

and
\[ \int_{-\infty}^{\infty} [-c_y I_r (c_2 \varphi_1 - b_2 \varphi_1^3) + c_y I_r \varphi_1 + f \sin \varphi_1] dt = c_y I_r \Delta \varphi. \] (53)

The third integral can be rewritten as
Using \( \Delta \phi = \phi(+\infty) - \phi(-\infty) \) yields

\[
M^L_1 = \int_{-\infty}^{+\infty} \left[ \sin \phi \left( c_2 \phi_x - k_2 \phi_x^2 - k_1 \phi^2 \right) \right] d\phi
\]

By the equation (48), we have

\[
\sin \phi(-\infty) = \frac{c_1 L}{f_1}, \quad \cos \phi(-\infty) = \sqrt{1 - \frac{c_1^2 L^2}{f_1^2}}.
\]

Substituting equation (57) into (56) yields

\[
M^L_1 = L \left[ \frac{\sqrt{f_1^2 - c_1^2 L^2} \cos \Delta \phi - 1 - c_1 L \sin \Delta \phi}{} \right].
\]

By the equations (53), (54) and (58), the Melnikov function can be given as

\[
M^L = -\frac{2\sqrt{2} \beta^2 \rho^2}{3 \sigma} \left[ y(1 - \sigma) - k_2 \rho^2 \phi_x^2 + c_2 \phi_x^2 \Delta \phi \right]
\]

\[
+ L \left[ \sqrt{f_1^2 - c_1^2 L^2} \cos \Delta \phi - 1 - c_1 L \sin \Delta \phi \right].
\]

In order to determine the existence of Silnikov-type homoclinic orbit, one must require that the Melnikov function \( M^L \) has a simple zero. Thus, the following expression is obtained

\[
2\sqrt{2} \beta^2 \rho^2 \left[ y(1 - \sigma) - k_2 \rho^2 \phi_x^2 \right]
\]

\[
- c_2 \phi_x^2 \Delta \phi - L \left[ \sqrt{f_1^2 - c_1^2 L^2} \cos \Delta \phi - 1 - c_1 L \sin \Delta \phi \right] = 0.
\]

The following condition must be satisfied for the orbit in \( W^u(p \varepsilon) \) to return to the basin of attraction of \( P \varepsilon \):

\[
\phi_{\min} < \psi_e + \Delta \phi + \mu_{\min} < \psi_e.
\]

Where \( m \) is an integer, \( \Delta \phi, \psi_e, \phi_{\min} \) and \( \phi_{\min} \) are respectively given by (41), (48) and (50).

**Numerical Simulation**

In this section, system (4) and averaged equation (5) are used to do numerical simulations for the single-pulse and multi-pulse chaotic dynamics. When the parameters of the system (5) are chosen as \( c_x = 0.05, c_y = 0.02, \sigma_1 = 0.8, \sigma_2 = 0.2, \)

\( f_1 = 5, k_1 = 2.2, k_2 = 3.8, k_3 = 1.9, k_4 = -2.7, k_5 = 3.2, k_6 = 1.9, f = 5 \), the single-pulse chaos occurs, as shown in Fig.3. Fig.3 (a) and (b) represent the phase portraits on the planes \( (x_1, x_2) \) and \( (x_3, x_4) \), and (c) and (d) indicate three-dimensional phase portraits in the phase places \( (x_1, x_3, x_4) \), \( (x_2, x_3, x_4) \).

186
Figure 3. The chaotic motions of the system (5) with $C_x = 0.05, C_y = 0.02, \sigma_1 = 0.8, \sigma_2 = 0.2, f_1 = 5, k_1 = 2.2, k_2 = 3.8, k_3 = 1.9, k_4 = 1.9, f = 5$

(a) phase portrait on plane $(x_1, x_2)$; (b) phase portrait on plane $(x_3, x_4)$; (c) phase portrait in three-dimensional space $(x_1, x_3, x_4)$; (d) phase portrait in three-dimensional space $(x_2, x_3, x_4)$.

By the relations of the parameters in system (4) and (5) (see [5] for more details), the values of the parameters in the system (4) can be obtained. Fig.4 indicates that the responses of the system (4) with the parameters $c_x = 0.05, c_y = 0.02,$

$\sigma_1 = 0.8, \sigma_2 = 0.2, f_1 = 5, \alpha_1 = 0.5, \beta_1 = 0.6, \alpha_3 = 0.625, \eta_2 = 7.21, \eta_3 = 5.67, \gamma_2 = 4.05, \gamma_3 = -0.74,$

$\Omega = 0.23, \Omega_1 = 0.46, \omega_x = 0.23, \omega_y = 0.23, f = 5, \epsilon = 0.05$ (corresponding to the case in Fig.3).

Based on the numerical simulations, we obtain the conclusions that when the chaotic motions occur in the averaged system, the motions of the suspended cables with parametric and external excitations are also chaos. Hence, the chaotic motions in the averaged equations can lead to the amplitude-modulated chaotic oscillations in the original system under certain conditions.

Conclusion

The global bifurcations and chaotic dynamics in the parametrically and externally excited suspended cable system are investigated by using the analytical and numerical approaches. For the averaged system, sufficient conditions for the existence of single-pulse and multi-pulse Silnikov-type homoclinic orbits which are created by the perturbation in a resonance are obtained by using the higher-dimensional Melnikov method and extended Melnikov method. It is well known that the chaotic motions in the averaged system can lead to the amplitude modulated chaotic oscillations in the original system under certain conditions. Numerical results have verified that the existence of chaos in the averaged system, and correspond chaotic motions also occur in the original system.

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References


