Weighted Multidimensional Scaling Localization of a Known Altitude Object Using TOA Measurements

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Abstract. This paper considers the problem of geolocating a target on the Earth surface whose altitude is known using the target signal time of arrival (TOA) measurements. The geolocation Cramer-Rao lower bound (CRLB) is derived and the performance improvement due to the availability of target altitude information is quantified. An weighted multidimensional scaling geolocation solution is developed. Its sensitivity to the target altitude error is also studied. Simulations verify the theoretical developments and illustrate the good performance of the proposed geolocation method.

Introduction

As a classical problem, passive source localization using time of arrival (TOA) measurements has gained considerable interests in various fields, such as radar, sonar, wireless communication and so on [1-3]. Localization techniques have been extensively investigated for deducing the target position from TOAs of the target signal received at spatially distributed receivers [2-4]. In particular, Wei et al. proposed in [4] a weighted multidimensional scaling (MDS) algorithm by using the statistical distributions of the measurements. It was shown to be able to reach the Cramer-Rao lower bound (CRLB) at small noise level and it performs better than the two-step weighted least square (TSWLS) method [4] under moderate measurement noise.

In this paper, we consider the passive geolocation of a source on the Earth surface using TOA measurements. The source altitude information can come from e.g. an altimeter [5-6] or simply the prior information that the source is on the ground. To exploit effectively the target altitude information, algebraic and iterative solutions (see e.g., [5], [6] and references therein) are available. But they were both developed for geolocating a known altitude object using the time difference of arrival (TDOA) and frequency difference of arrival (FDOA) measurements.

The study in this paper begins with mathematically formulating the TOA-based geolocation problem and deriving the target position Cramer-Rao lower bound (CRLB). The contribution of the source altitude information to improving the geolocation accuracy is investigated. By considering the Earth center as a virtual receiver and imposing a prior distribution on the target altitude information, the target geolocation problem is cast into an MDS framework. The sensitivity of the geolocation accuracy to the error in the target altitude information is quantitatively analyzed. Simulations corroborate the theoretical developments and demonstrate better performance of the proposed MDS geolocation technique over benchmark methods.

Problem Formulation

Consider an unknown target located on the surface of Earth with the true position $\mathbf{u}^o = [x^o, y^o, z^o]^T$. There are $M$ sensors at $s_i = [x_i, y_i, z_i]^T$, $i = 1, 2, \ldots, M$, where $[^T]$ denotes the matrix transpose operation and $M \geq 3$.

With the target altitude information, we have
\[ u^o^T u^o = R^2 \]  
where \( R \) is equal to the sum of the target altitude and the local Earth radius [6-7]. The true distance between the target and sensor \( i \) is 
\[ r^o_i = \| u^o - s_i \| = c t^o_i = r_i - \Delta r_i \]  
where \( \| \cdot \| \) denotes the Euclidean norm. \( c \) is the signal propagation speed, \( t^o_i \) is the true TOA, \( r_i \) is the range measurement, and \( \Delta r_i \) is the measurement noise assumed to be zero-mean Gaussian distributed. For the sake of simplicity, the true range vector is defined as \( r^o = [r_1^o, r_2^o, \ldots, r_M^o]^T \). We have \( r = r^o + \Delta r \), where \( r = [r_1, r_2, \ldots, r_M]^T \), and \( \Delta r = [\Delta r_1, \Delta r_2, \ldots, \Delta r_M]^T \). The covariance matrix of the TOA measurement error vector \( \Delta r \) is \( Q_r \).

**CRLB Analysis**

The Cramer-Rao lower bound (CRLB) gives the lowest possible estimation covariance matrix for any unbiased estimator of deterministic parameters [7-8]. From the previous section, we have that the unknown is the target position \( u^o \). Note from (1) that \( u^o \) is equality-constrained and its CRLB is therefore a constrained one, which will thus be denoted as \( \text{CCRLB}(u^o) \). It is equal to [5-6]

\[ \text{CCRLB}(u^o) = J^{-1} - J^{-1} F (F^T J^{-1} F)^{-1} F^T J^{-1} \]  
Here, \( F \) is the Jacobian of the constraint (1), which is

\[ F = u^o \]  

As shown in [6], \( J^{-1} \) is equal to

\[ J^{-1} = \left( \frac{\partial r^o}{\partial u^o} \right)^T Q_r^{-1} \left( \frac{\partial r^o}{\partial u^o} \right) \]  

where

\[ \frac{\partial r^o}{\partial u^o} = \begin{bmatrix} \rho_{u^o,s_1} & \rho_{u^o,s_2} & \cdots & \rho_{u^o,s_M} \end{bmatrix}^T \]  
and \( \rho_{u^o,s_i} = (u - s_i)/\|u - s_i\|, \ i = 1, 2, \ldots, M \).

\( J^{-1} \) is indeed the CRLB of the target position \( u^o \) when its altitude information is not available [7-8]. Define \( J^{-1} = \text{CRLB}(u^o) \) for sake of clarity. As a result, we can observe from (3) that the utilization of the target altitude information via (2) can in effect lead to improved performance in terms of reduced target geolocation CRLB.

**Geolocation Algorithm and Analysis**

**Algorithm**

Let us consider the Earth center as a virtual receiver at the origin such that the equality constraint on the target position in (1) becomes a noiseless TOA measurement equation. As a result, the MDS framework can be employed for TOA-based target geolocation.
Mathematically, we denote \( r_0^o = \| u^o - s_0 \| = R \), where \( s_0 = [0, 0, 0]^T \) is the Earth center. Augmenting the true TOA vector, we have \( r_e^o = [r_0^o, r_1^o, \ldots, r_M^o]^T \) and the TOA measurement covariance matrix now becomes \( Q_{re} = \text{diag}(q, Q) \), where \( q \) is the variance of the known target altitude and \( q = 0 \).

Now, we can obtain the source position estimation via following an approach similar to that in [4]. Furthermore, according to [4], the weighted MDS algorithm can reach the CRLB at small noise level. And the method details and the performance analysis of the weighted MDS algorithm are omitted there, they are can be find in [4].

**Effect of Altitude Error**

We shall investigate the impact of the uncertainty in the target altitude information on the geolocation accuracy. Different from the errors in the TOA measurements, which are assumed to be Gaussian random, the altitude error, denoted as \( \Delta h \), is generally unknown but deterministic.

Considering the line-of-sight transmission scene, if the sensor acceptance range of target signal is 350km, from the geometries, the target altitude variation range is \( 0 \sim 10 \) km. We can assume the \( \Delta h \) is a uniform distribution random, and its mean-square is \( q = (h_{\text{max}} - h_{\text{min}})^2 / 12 \), where \( h_{\text{min}} \) and \( h_{\text{max}} \) is the minimum and the maximum of the target altitude, respectively.

Moreover, the [5] have a conclusion that if the following condition is fulfilled

\[
\Delta h \leq \frac{1}{R} \sqrt{u^o (G_s^o W G_s^o)^{-1} u^o} \tag{7}
\]

there is

\[
E(\Delta u \Delta u^T) \leq (G_s^o W G_s^o)^{-1} \tag{8}
\]

where \( E(\Delta u \Delta u^T) \) is the source localization error covariance matrix with altitude imprecise.

Rewrite (7), we have

\[
\Delta h \leq \frac{1}{R} \sqrt{\frac{u^o (G_s^o W G_s^o)^{-1} u^o}{(G_s^o W G_s^o)^{-1}}} = \frac{1}{R} \sqrt{\frac{R^2}{(G_s^o W G_s^o)^{-1}}} = \sqrt{(G_s^o W G_s^o)^{-1}} \tag{9}
\]

Further, according [5], we find

\[
G_s^o = -2 \begin{bmatrix}
  s_1^T - u^oT \\
  s_2^T - u^oT \\
  \vdots \\
  s_M^T - u^oT
\end{bmatrix},
B = 2 \begin{bmatrix}
  r_1 & 0 & 0 & \cdots & 0 \\
  0 & r_2 & 0 & \cdots & 0 \\
  \vdots & \vdots & \vdots & \ddots & \vdots \\
  0 & 0 & 0 & \cdots & r_M
\end{bmatrix}
\]

\[
W = (BQ, B^T)^{-1} \tag{11}
\]

So,

\[
(G_s^o W G_s^o)^{-1} = J^{-1} = \text{CRLB} \tag{13}
\]

where \( \text{CRLB} \) is the theoretic location error covariance matrix without target altitude information using TOA measurements only.

This means that when the target altitude is known imprecisely but its error satisfies (9), namely, if the altitude error is not bigger than the source localization mean square error only using TOA measurements, exploring source altitude information can still improve the geolocation performance.
over the case where only TOA measurements are utilized. However, the altitude error may significantly degrade the geolocation accuracy, if the condition (9) is violated.

Simulations

Consider $M = 8$ receivers whose true positions are summarized in Table 1. The target is located at $(104.0381\degree E, 30.7650\degree N)$ with an altitude of 0m. The covariance matrices of the TOA measurement errors are set to be $Q_d = \sigma_d^2 R$, $\sigma_d$ is the TOA measurement noise. $R$ is a $(M-1) \times (M-1)$ matrix with the diagonal elements being equal to 1 and the off-diagonal elements all equaling to 0.5.

The geolocation accuracy of the proposed MDS solution is quantified by the root mean square error (RMSE), defined as

$$RMSE(u) = \sqrt{\frac{1}{L} \sum_{l=1}^{L} \|u_l - u^*\|^2}$$

where $u_l$ is the target position estimate at the $l$th ensemble run, and $L = 5000$ is the total number of ensemble runs.

<table>
<thead>
<tr>
<th>receiver no. $i$</th>
<th>longitude(\degree E)</th>
<th>latitude(\degree N)</th>
<th>altitude(m)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>104.0214E</td>
<td>30.6535N</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>104.0250E</td>
<td>30.6800N</td>
<td>0</td>
</tr>
<tr>
<td>3</td>
<td>104.0486E</td>
<td>30.6796N</td>
<td>0</td>
</tr>
<tr>
<td>4</td>
<td>104.0555E</td>
<td>30.6476N</td>
<td>0</td>
</tr>
<tr>
<td>5</td>
<td>104.0213E</td>
<td>30.6690N</td>
<td>0</td>
</tr>
<tr>
<td>6</td>
<td>104.0350E</td>
<td>30.6807N</td>
<td>0</td>
</tr>
<tr>
<td>7</td>
<td>104.0555E</td>
<td>30.6618N</td>
<td>0</td>
</tr>
<tr>
<td>8</td>
<td>104.0381E</td>
<td>30.6507N</td>
<td>0</td>
</tr>
</tbody>
</table>

For the purpose of comparison, we simulate a benchmark technique, referred to as the Lagrange multiplier (LM) method in [5], and iterative Lagrange multiplier (ILM) method in [6].

Figure 1 compares the geolocation accuracy of the LM, ILM and proposed MDS method solutions as a function of the standard deviation of the TOA measurement noise $\sigma_d$. An altitude error $\Delta h$ of 0m is assumed.

In the last experiment, we investigate the effect of target altitude error. The results are summarized in Figure 2-3, where as a function of the target altitude error $\Delta h$, the geolocation RMSEs of the proposed algorithm in considerations are contrasted with respect to the theoretical results given in (3). In Figure 3, we set $\sigma_d = 10^{-1}$ m.
Figure 3. Effect of errors in the known target altitude.

Also included in Figure 1-3 are the associated target geolocation CRLBs in (3) \((\text{CCRLB}(\mathbf{u}^v))\) and the CRLBs of the target position \((\text{CRLB}(\mathbf{u}^v))\) when the altitude information is absent.

We obtain the following observations from Figs. 1-3:

1. Comparing \(\text{CCRLB}(\mathbf{u}^v)\) and \(\text{CRLB}(\mathbf{u}^v)\) reveals that exploring the target altitude information can significantly improve the target geolocation accuracy.

2. Both the LM, ILM and proposed MDS methods are able to attain the CRLB accuracy under small noise conditions. But the proposed MDS appears to be more robust to larger noise levels.

3. In this simulation, the geolocation RMSE from simulations matches the theoretical value well. This justifies the validity of the analysis in Section 4.

Conclusion

This work investigated geolocating a target on the Earth surface from TOA measurements. CRLB analysis showed that the use of target altitude information can improve the target geolocation accuracy. An MDS solution was developed. It can reach the CRLB accuracy under small Gaussian noise and it was shown to be able to outperform the benchmark techniques at relatively large noise levels.

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References


