Hierarchical Bayesian Estimation of System Parameters from Dynamic Responses

Qing YANG and Kun ZHANG*

School of Civil and Safety Engineering, Dalian Jiaotong University, Dalian, 116028, China

*Corresponding author

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Abstract. A hierarchical Bayesian approach is proposed for estimating system parameters by directly taking dynamic responses as the fixed target. The basic theories of hierarchical Bayesian model are first introduced, and then the estimation process of system parameters is illustrated. A mass-spring system of eight degrees of freedom is studied to validate the proposed method, and the effects of modeling errors, incompleteness of dynamic response data and measurement noise on the identification results are numerically investigated. The research results show that the proposed method can accurately identify physical parameters of the system with random modeling errors and measurement noise from only several dynamic responses of the system. This method can provide a new way for parameter estimation of the system with modeling errors, incompleteness of dynamic response data and serous pollution of measurement noise.

Introduction

Parameter estimation from system vibration information and structural model updating, fault diagnosis or damage identification based on the estimated system parameters have been receiving more and more attentions from researchers in the last few decades. However, the nature of the system model and observed data has strong uncertainty under the influence of some factors such as the observation noise, modeling error and environmental factors etc., which causes the estimation of system parameters become a kind of uncertain problems[1].

In order to overcome the negative effects of uncertainty factors on the estimation accuracy of system parameters, some methods based on Bayesian models have been proposed in literatures[2-5]. In these approaches, the Bayesian asymptotic is used to get the posterior joint probability distribution of the system parameters and their optimal value, thus realizing the Bayesian estimation of unknown system parameters. However, the available Bayesian approaches can successfully predict the estimation uncertainties of the estimating parameters (e.g., structural stiffness, damping or mass), but they do not consider the inherent variability of these parameters due to different sources of uncertainties such as changing ambient temperature, temperature gradient, wind speed, and traffic load[6]. Compared with the traditional Bayesian approaches, the hierarchical Bayesian approach has better Potential applications for the parameter estimation of the systems with complex influence of external environment uncertainty since the hierarchical Bayesian models provide an account of Bayesian inference in a hierarchically structured hypothesis space. Hierarchical Bayesian approach was first used to identify civil structural systems under changing ambient/environmental conditions by Iman Behmanesh et al. in 2015[6]. In their hierarchical Bayesian model frame, the structural modal parameters (frequency and vibration mode) were used as a fixed target to realize the estimation of structural physical parameters. However, the existing studies show that there are few insufficient in the parameter identification methods based on frequency and vibration model, such as lower frequencies and vibration modes are not sensitive to damage, while the measurement errors of higher order frequencies and vibration modes are often large, especially the mode measurement errors and errors caused by incomplete modal mix. These lead poorer identification precision of modal parameters themselves[7,8]. Therefore, the hierarchical Bayesian method with the modal parameters as the fixed target has some limitations in the application.
In this paper, the dynamic responses of systems are directly taken as the fixed target to construct the parameter estimation framework of hierarchical Bayesian model. Firstly, the basic theories of hierarchical Bayesian model are firstly introduced, and the relative estimation process of system parameters from dynamic responses is illustrated. After that, a mass-spring system of eight degrees of freedom is studied to validate the proposed method, and the effects of modeling errors, incompleteness of dynamic response data and measurement noise on the identification results are numerically investigated. Finally, the conclusions of this work are summarized.

**Basic Theories of Hierarchical Bayesian Models Based on Dynamic Responses**

In the hierarchical Bayesian model based on dynamic responses, the unknown parameters of the system are assumed to obey the Gaussian normal distribution $\theta \sim N(\mu_\theta, \Sigma_\theta)$, and the error function vector is defined as the error between the calculated and the observed dynamic responses from chosen measurements. This can be represented by a multivariate Gaussian distribution as

$$e_i = D - \tilde{D} \sim N(u_i, \Sigma_e)$$

(1)

where $D$ and $\tilde{D}$ represent the calculated and the observed dynamic responses of the system, respectively. $u_i$ is the mean of the error vectors, and $\Sigma_e$ represents the co-variance matrix for the error function.

The posterior probability distribution of the unknown parameters can be expressed as

$$p(\theta_\theta, \Sigma_\theta, u_e, \Sigma_e|\tilde{D}) \propto P(\tilde{D}|\theta\theta, u_e, \Sigma_e)p(\theta|\mu_\theta, \Sigma_\theta)p(\mu_\theta, \Sigma_\theta, u_e, \Sigma_e)$$

(2)

Different from the traditional Bayesian method, super parameters are introduced in the prior probability distribution of multistage Bayesian model. $\mu_\theta, \Sigma_\theta, u_e, \Sigma_e$ in Eq.(2) all are the over-parameters in this distribution. Before determining the probability distribution, the transcendental probability distribution $p(\theta, \Sigma_\theta, u_e, \Sigma_e)$ of the over-parameters is determined first, and then it constitutes the posterior probability distribution of multistage Bayes together with prior probability distribution and likelihood function.

In the case of having $N_t$ independent data sets, the joint posterior probability distribution function (PDF) can be stated as

$$p(\Theta, \mu_\theta, \Sigma_\theta, u_e, \Sigma_e|\tilde{D}) \propto \prod_{i=1}^{N_t} P(\tilde{D}|\theta_i, u_e, \Sigma_e)p(\theta_i|\mu_\theta, \Sigma_\theta)p(\mu_\theta, \Sigma_\theta, u_e, \Sigma_e)$$

(3)

where $\Theta = \{\theta_1, \ldots, \theta_{N_t}\}$. It is often reasonable to assume no correlation between these parameters and therefore the co-variance matrix $\Sigma_\theta$ can be presented as a diagonal matrix $\Sigma_\theta = Diag(\sigma_\theta^2, \sigma_\theta^2, \ldots, \sigma_\theta^2, \ldots, \sigma_\theta^2)$, with $N_\theta$ is the number of the unknown parameters in the vector of $\theta$. Note that the formulations can be extended for correlated system parameters by estimating all components of the full co-variance matrix $\Sigma_\theta$. An inverse Gamma probability distribution is assumed for the prior probability of the $\sigma_\theta^2$, and that is

$$p(\sigma_\theta^2) = \text{Inverse Gamma}(\alpha, \beta)$$

(4)

where $\alpha$ and $\beta$ can be taken identically for all the unknown structural parameters. Based on the considered priors, the joint posterior probability distribution of all the updating parameters can be stated as

$$p(\Theta, \mu_\theta, \Sigma_\theta, u_e, \Sigma_e|\tilde{D}) \propto$$

393
\[
\frac{1}{\prod_{\beta=1}^{N_{\beta}} N_{\beta}} \prod_{p=1}^{N_p} (\sigma_{\phi p}^2)^{-1} e^{-\frac{1}{2} \sum_{i=1}^{N_i} \left( \theta_i - \bar{D}_i, u_{e i}, \Sigma_e \right) \left( \theta_i - u_{\phi} \right)^T \Sigma_e^{-1} \left( \theta_i - u_{\phi} \right)}
\]

where

\[
J(\theta, \bar{D}_i, u_e, \Sigma_e) = (e_i - u_{e i})^T \Sigma_e^{-1} (e_i - u_{e i})
\]

\[
p(\theta | u_\theta, \Sigma_\theta, u_e, \Sigma_e, \bar{D}) \propto \exp \left\{ -J(\theta, \bar{D}_i, u_{e i}, \Sigma_e) - \sum_{p=1}^{N_p} \left( \theta_{p\theta} - u_{\phi p} \right)^2 \right\}
\]

\[
p(u_\theta | \Sigma_\theta) = N \left( \frac{1}{N_t} \sum_{i=1}^{N_t} \theta_i, \frac{1}{N_t} \Sigma_\theta \right)
\]

\[
p(\Sigma_e | \theta, \Sigma_\theta, u_e) = \text{InverseGamma} \left( \frac{N_t}{2} + \alpha, \frac{1}{2} - \frac{1}{2} \sum_{i=1}^{N_t} (\theta_{p\theta} - u_{\phi p})^2 \right)
\]

\[
p(u_{\phi} | \Sigma_e) = N \left( \frac{1}{N_i} \sum_{i=1}^{N_i} e_i, \frac{1}{N_i} \Sigma_e \right)
\]

\[
p(\Sigma_e | \theta, \Sigma_\theta, u_e) = \text{InverseWishart} \left( \alpha, I(N_e) + \sum_{i=1}^{N_i} (e_i - \mu_e)(e_i - \mu_e)^T, N_i + N_e \right)
\]

The most common technique to solve Eq. (5) is the Gibbs Sampler, in which samples are generated from the full conditional probability distribution of each parameter until convergence is reached. Convergence is achieved when the changes in statistics of generated samples becomes smaller than a prescribed threshold. After convergence, the MAP estimates of all the updating parameters provide the global maxima of the posterior joint PDF of Eq.(7). The full conditional posterior probability distributions of all the unknown parameters are presented in Eqs.(7)-(11).

It can be observed that the full conditional probability distributions of all the unknown parameters except \( \theta_i \) are standard distributions. The posterior joint probability distribution of unknown parameters can be accurately estimated if an adequate number of samples is generated. Generating samples from Eqs. (8)-(11) is trivial due to their known distribution functions. However, when samples are generated from the conditional probability distributions of \( \theta_i \) by using Eq. (7), it is required to use advanced sampling techniques such as Metropolis-Hasting [6]. Repeat the process until get the smooth Markov chain. The proposed distribution is based on the random distribution of the theoretical values of the unknown parameters in the process.

**Numerical Simulation**

**Model Description**

A mass-spring system of eight degrees of freedom [9] is taken as an example to validate the proposed algorithm. The mass of block 1 is 559.3 g, while that of block 2-7 is 419.4 g. The stiffness value of all the springs is 56.7 kN/m. Random errors of 20% level are included in the stiffness coefficients of each spring to simulate the uncertainties of modeling error. In order to investigate the uncertainty of incompleteness of dynamic response data, only the accelerations of blocks 2, 5 and 7 are arbitrarily taken as the measured dynamic responses in estimating the unknown parameters. Moreover, two
perturbation amplitudes of 5% and 20%, indicating that two levels of measurement noise are contained in the measured data, are simulated to investigate the effect of the observation noise uncertainties on the identification accuracy.

The system is assumed to be excited at the block one (as shown in Fig.1) by a random Gaussian white noise. The sampling frequency and the sampling time of structural dynamic responses are 500 Hz and 1 s, respectively. The identification accuracy is represented by the relative error between the identified stiffness coefficients and the true ones as

$$ e_j = \frac{k_j^{id} - k_j^{tr}}{k_j^{tr}} \times 100\% , \ j=1,2,...,7 \quad (12) $$

where $k_j^{id}$ and $k_j^{tr}$ are the jth identified and true stiffness coefficients respectively.

Figure 1. The schematic of the eight degrees of freedom mass-spring system.

**Result Analysis**

Table 1 listed the true random stiffness coefficients of the system and their identified values with different level of noise. Fig.2 illustrates the relative errors between the identified stiffness coefficients and the true ones in the three cases listed in Table 1. As shown in Table 1 and Fig.2, for all the three cases with or without measurement noise, the identified stiffness coefficients all are close to their true values and the absolute maximum relative errors of all the three cases are below 3%. These results indicate the proposed method can accurately identify physical parameters of the system with random modeling uncertainties from only several dynamic responses of the system. The identification errors are mainly caused by the random nature of the Bayesian models, while the uncertainty of measurement noise has little negative influence on the estimating accuracy. The proposed hierarchical Bayesian approach has good robustness to measurement noise.

<table>
<thead>
<tr>
<th>No. of elements</th>
<th>True stiffness (kN/m)</th>
<th>Identified stiffness (kN/m)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>without noise</td>
<td>with 5% noise</td>
</tr>
<tr>
<td>1</td>
<td>66.2</td>
<td>66.3</td>
</tr>
<tr>
<td>2</td>
<td>57.8</td>
<td>57.3</td>
</tr>
<tr>
<td>3</td>
<td>48.6</td>
<td>48.5</td>
</tr>
<tr>
<td>4</td>
<td>55.1</td>
<td>54.0</td>
</tr>
<tr>
<td>5</td>
<td>62.4</td>
<td>60.7</td>
</tr>
<tr>
<td>6</td>
<td>53.6</td>
<td>54.0</td>
</tr>
<tr>
<td>7</td>
<td>60.8</td>
<td>61.8</td>
</tr>
</tbody>
</table>
Figure 2. The relative errors between the identified stiffness coefficients and the true ones.

Summary

Based on the existing hierarchical Bayesian algorithm, this paper proposes a hierarchical Bayesian system parameter estimation approach from the dynamic responses of the system in time domain. The validity of the proposed method is numerically verified by an eight-degree-of-freedom mass-spring system. The research results show that the proposed method can accurately identify physical parameters of the system with random modeling errors from only several dynamic responses of the system, and has good robustness to measurement noise uncertainty. This method can provide a new way for parameter estimation of the system with modeling errors, incompleteness of dynamic response data and serous pollution of measurement noise. Of course, further investigation on the performance and the validity of approach on identifying experimental or real engineering system are still on the way.

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References


