The Vertex-Degree-Distance of the Extended-Stars
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Abstract. The degrees-distance is a graph parameter related to the vertex degree and distance of the graph, which is a variation of the Wiener Index. In this paper, we study the extremal values of the vertex-degree-distance of the extended-stars and obtain the distribution of the maximum (vertices) and the minimum (vertices) of the vertex-degree-distance of the extended-stars.

Introduction

In chemical graph theory, the concept of topological index is introduced to characterize the structural properties of molecules, while the topological index of the distance between vertices of graph plays an important role in describing the molecular graph and establishing the relationship between molecular structure and features. It is widely used to predict the physicochemical properties and biological activity of compounds.

The Wiener index\(^1\) is an important topological index of the distance between vertices of a graph. It was proposed by Wiener in 1947 to investigate the relationship between the boiling point and the molecular structure of alkanes. It refers to the sum of the distances between all vertex pairs in a graph. And after that, many variants of Wiener index were proposed according to the vertex degree and the distance\(^2-8\). The degree-distance of graphs was introduced by Dobrynin and Kochetova in 1994\(^2\), and it extended the research area of parameters of graphs related to the Wiener index.

In this paper, we study the extremal values and rankings of the vertex-degree-distance of the extended-stars.

Definitions and Notation

Let \(G\) be a connected simple graph. \(W(G) = \sum_{u,v \in V(G)} d(u,v)\) is called the Wiener index of the graph \(G\), where \(d(u,v)\) is the distance between vertices \(u\) and \(v\) of \(G\).\(^1\) \(\forall v \in G\), call \(d(v) = \sum_{u \in V(G)} d(u,v)\) (denoted as \(d'(v)\)) the degree-distance of the vertex \(v\), where \(d(v)\) is the degree of the vertex \(v\), and call \(\sum_{v \in V(G)} d'(v)\) (denote as \(D'(G)\)) the degree-distance of the graph \(G\).\(^2\)

An \((m, n)\)-extended-star (extended-star, denoted as \(ES_{mn}\)) is a graph obtained by inserting \(n-1\) \((n \geq 2)\) vertices on each pendent edge of the star \(S_{1,m}\) \((m \geq 3)\) (see figure 1). The distance between a vertex and the center is called the level of the vertex.
Main Results

Theorem 1. The vertex-degree-distance of the center \(v_0\) of the extended-star \(ES_{mn}\) is
\[
d'(v_0) = \frac{n(n+1)m^2}{2}.
\]

Proof. From definition 2, we get
\[
d'(v_0) = d(v_0) \sum_{u \in V(G)} d(u, v_0) = m \sum_{j=1}^{n} jm = \frac{n(n+1)m^2}{2}.
\]

Theorem 2. The vertex-degree-distance of the \(j\)th level vertex \(v_{ij}\) \((i = 1, 2, \ldots, m; 0 < j < n)\) of the extended-star \(ES_{nm}\) is
\[
d'(v_{ij}) = 2j^2 + 2(m-2)nj + mn(n+1).
\]

Proof. It is clear that \(d'(v_{ij}) = d'(v_{ij}) = \cdots = d'(v_{im})\) for \(0 < j < n\). So, we need count \(d'(v_{ij})\) only. Denote that (see Fig. 2)
\[
P_1(v_{ij}) = v_{1i}v_{1i-1} \cdots v_{1i+1},
\]
\[
P_2(v_{ij}) = v_{1i-1}v_{1i-2} \cdots v_{1i}v_0,
\]
\[
P_3(v_{ij}) = \{v_{st} | s = 2, 3, \ldots, m; t = 1, 2, \ldots, n\}.
\]

Then
\[
d'(v_{ij}) = d(v_{ij}) \left( \sum_{x \in V(P_1(v_{ij}))} d(v_{ij}, x) + \sum_{x \in V(P_2(v_{ij}))} d(v_{ij}, x) + \sum_{x \in V(P_3(v_{ij}))} d(v_{ij}, x) \right)
\]
\[
= 2 \left( \sum_{l=j+1}^{n} d(v_{ij}, v_{il}) + \sum_{l=0}^{j-1} d(v_{ij}, v_{il}) + (m-1) \sum_{l=1}^{n} d(v_{ij}, v_{im}) \right)
\]
\[
= 2 \left( \sum_{l=j+1}^{n} (l - j) + \sum_{l=0}^{j-1} (j - l) + (m-1) \sum_{l=1}^{n} (l + j) \right)
\]
\[
= 2j^2 + 2(m-2)nj + mn(n+1).
\]
Theorem 3. The vertex-degree-distance of the \( n \)th level vertex \( v_{in} \) \((i=1,2, \ldots, m)\) of the extended-star \( ES_{nm} \) is \( d'(v_{in}) = \frac{1}{2}(3m-2)n^2 + mn \).

Proof. It is clear that \( d'(v_{in}) = d'(v_{2n}) = \cdots = d'(v_{mn}) \). So, we need count \( d'(v_{in}) \) only. From the proving process of Theorem 2, we can get

\[
d'(v_{in}) = d(v_{in}) \left( \sum_{x \in V(P_2(v_{in}))} d(v_{in}, x) + \sum_{x \in V(P_3(v_{in}))} d(v_{in}, x) \right)
\]

\[
= \sum_{j=0}^{n-1} d(v_{in}, v_{in}) + (m-1) \sum_{j=1}^{n} d(v_{in}, v_{jn})
\]

\[
= \sum_{j=0}^{n-1} (n-l) + (m-1) \sum_{l=1}^{n} (d+n)
\]

\[
= \frac{1}{2}(3m-2)n^2 + mn.
\]

Theorem 4. The degree-distances of the \((m,n)\)-extended-star is as follows:

\[
D'(ES_{mn}) = \frac{mn}{3} \left(2(3m-2)n^2 + 3mn + 1\right).
\]

Proof. From theorem 2, theorem 3 and theorem 4, we get

\[
D'(ES_{mn}) = \sum_{v \in V(ES_{mn})} d'(v) = d'(v_0) + m \sum_{j=1}^{n-1} d'(v_{ij}) + md'(v_{in})
\]

\[
= \frac{mn}{3} \left(2(3m-2)n^2 + 3mn + 1\right).
\]

Theorem 5. The extremal values (vertices) of the vertex-degree-distances of the \((m,n)\)-extended-star is as follows:

\[
\max_{v \in V(ES_{mn})} \{d'(v)\} = d'(v_0), \quad \min_{u \in V(ES_{mn})} \{d'(u)\} = d'(v_{in}).
\]

Proof. (1) It is clear that

\[
d'(v_{i,n-1}) = \max_{1 \leq j \leq n-1} \{d'(v_{ij})\} = mn(n+1) + 2(m-1)j \big|_{j=n-1} = mn^2 + (3m-2)n - 2m + 1.
\]

And

\[
d'(v_0) - d'(v_{i,n-1}) = \frac{1}{2}(m(m-2)n^2 + (m^2 - 6m + 4)n + 2(2m-1)) > 0.
\]
\[ d'(v_0) - d'(v_{in}) = \frac{1}{2}((m-1)(m-2)n^2 + mn(m-1)) > 0. \]

So, \( d'(v_0) = \max_{v \in V(ES_{mn})} \{d'(v)\} \).

(2) It is clear that
\[
d'(v_{in}) = \min_{v \in V(ES_{mn}) \setminus \{v_0\}} \{d'(v)\} = 2j^2 + 2(m-2)nj + mn(n+1) \big|_{j=n-1} = mn^2 + (3m-4)n + 2.
\]

And \( d'(v_{in}) - d'(v_{in}) = \frac{1}{3}(2n^2 + (5m-12)n + 6) \). Let \( f(x, y) = 2y^2 + (5x-12)y + 6 \). Then, \( \frac{\partial f}{\partial x} = 5y > 0 \) for \( y \in [2, +\infty) \). So,
\[
f(2, y) = \min_{x \geq 2} \{f(x, y)\} = 2y^2 - 2y + 6 > 0 \ (y \geq 2).
\]

That is, \( d'(v_{in}) - d'(v_{in}) = \frac{1}{3} f(m, n) > 0 \), \( d'(v_{in}) < d'(v_{in}) \). Thus
\[
\min_{u \in V(ES_{mn})} \{d'(u)\} = d'(v_{in}).
\]

**Conclusions**

In this paper, we get the distribution of the extremal values (vertex) of the extended-stars by studying the vertex-distance of the extended-stars. The vertex distance gets the maximum at the vertex (the center) with the maximum vertex-degree and the minimum at the vertex (the leaf) with the smallest degree. These results reveal that the distribution of the extremal values (vertex) of the vertex-distance of a graph is related to the size of the vertex-degrees and the distance from the center.

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**References**


