Adjacent Vertex Distinguishing Edge-colorings of the Lexicographic Product of Special Graphs

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Abstract. A class of special graphs \( \Omega_n \) including wheels, fans and stars is defined. Afterwards, the adjacent vertex distinguishing edge-coloring of lexicographic product \( H \circ G \) of graph class \( \Omega_n \) and any graph \( G \) is studied, and gives an upper bound of the chromatic number of coloring. For special \( H \), the exact value of the adjacent vertex distinguishing edge-coloring of \( G[H] \) is obtained. In this paper, we prove that the chromatic number of adjacent vertex distinguishing edge-coloring of lexicographic product \( G[H] \) for any two graphs \( G \) and \( H \) is equal to the graph class \( \Omega_n \).

Introduction

Let \( G \) be a finite undirected simple connected graph, use \( V(G) \) and \( E(G) \) to denote the vertex set and edge set of a graph \( G \), respectively. And \( \Delta(G) \) denotes the maximum degree of a graph \( G \).

We refer to the books [1-2] for graph theory terminology and notation not defined in this paper.

**Definition 1.1** (Bondy et al.[1]) Let \( C = \{0,1,\cdots,k-1\} \) be a set of colors, edge-colorings \( \sigma \) of \( G \) is from \( E(G) \) to \( C \) the mapping, if satisfying adjacent edge-colorings have different colors, the minimum number of colors required for an edge-colorings of \( G \) is denoted by \( \chi'(G) \).

**Definition 1.2** (Zhang et al.[2]) Let \( \sigma \) be a edge-coloring of \( G \), and use \( S_\sigma(u) \) denotes the set of colors of all edges incident with \( u \), then \( S_\sigma(u) = \{\sigma(uv) | uv \in E(G)\} \). If for any \( uv \in E(G) \), have \( S_\sigma(u) \neq S_\sigma(v) \), then \( \sigma \) is the adjacent vertex distinguishing, the minimum number of colors required for an adjacent vertex distinguishing edge-coloring of \( G \) is adjacent vertex distinguishing edge chromatic number of \( G \) is denoted by \( \chi'_a(G) \).

In Definition 1.2, let \( S_\sigma(u) \) be a color set of the coloring \( \sigma \) with a vertex \( u \), and \( \overline{S_\sigma(u)} \) is a complement of \( S_\sigma(u) \). By Definition 1.2, it is easy to get the Lemma as follows.

**Lemma 1.1** Let \( G \) be a simple connected graph of order at least 3. If \( G \) has two adjacent vertices of maximum degree, then \( \chi'_a(G) \geq \Delta(G) + 1 \).

The definition of the lexicographic product \( G[H] \) in graph \( G \) is as follows:
Definition 1.3 (Bondy et al.[1]) The lexicographic product of two simple graphs $G$ and $H$ is the graph $G[H]$ with vertex set $V(G) \times V(H)$ of simple graphs, in which $(u,v)$ is adjacent to $(u',v')$ if and only if either $uu' \in E(G)$, or $u = u'$ and $vv' \in E(H)$.

In [2-7], Bondy and Zhang et al. discuss the adjacent vertex distinguishing edge-coloring of complete graphs, cycles, trees, grid graphs, hypercube, and the special graph of the lexicographic product, the cartesian product and double graphs. In this paper, we study the adjacent vertex distinguishing edge coloring of the lexicographic product of graph class $\Omega_n$, including wheels, fans, and stars, in which the construction methods of graph class $\Omega_n$ are as follows:

Let $G$ be a vertex set is $V(G) = \{u_1, u_2, \cdots, u_n\}$, and $\chi'(G) \leq n$. Use $\Omega_n$ to denote the set of all orders of $n$ simple graphs $G$ satisfying the following conditions (A): $G$ has a $n$-proper edge coloring $\sigma$, and the set of the coloring $\sigma$ is $\{A_1, A_2, \cdots, A_n\}$, we have

i) The set $\{\overrightarrow{S}_\sigma(u_1), \overrightarrow{S}_\sigma(u_2), \cdots, \overrightarrow{S}_\sigma(u_n)\}$ exist the system of distinct representatives, 

ii) For any edges $e(u, v) \in E(G)$ we have $\overrightarrow{S}_\sigma(u), -\{A_{\sigma(u)}\} \neq \overrightarrow{S}_\sigma(v) -\{A_{\sigma(v)}\}$, where $|\overrightarrow{S}_\sigma(u)| \geq 2$, $i, j = 1, 2, \cdots, n$.

Specifically, when $G$ is an empty graph of order $n$, let $\chi'(G) = 0$.

For any graphs of $\Omega_n$, by using the join operation of graphs, a special class of graphs can be obtained $\Omega_n$, $\Omega_n = \{G + K_1 | G \in \Omega_{n-1}\}$, where $G + K_1$ denoted join graph (Bondy et al.[1]) of a graph $G$ and a trivial graph.

By the definition of $\Omega_n$, any graphs $G$ of $\Omega_n$ has the unique maximum degree vertex and the maximum degree is $n-1$. Obviously, wheel $W_n$ of order $n \geq 5$, fan $F_n$ and star $S_n$ are elements of $\Omega_n$, and we have the following Lemma:

Lemma 1.2 For any graph $G \in \Omega_n$, we have $\chi'_d(G) = n - 1$.

Proof. Let $G = G_0 + K_1$, where $G_0 \in \Omega_{n-1}$, $V(K_1) = \{u_0\}$, $V(G_0) = \{u_1, u_2, \cdots, u_{n-1}\}$, $\sigma$ be a $(n-1)$-proper edge coloring of $G_0$ satisfying condition (A), $A_1, A_2, \cdots, A_{n-1}$ is system of distinct representatives of $\{\overrightarrow{S}_\sigma(u_1), \overrightarrow{S}_\sigma(u_2), \cdots, \overrightarrow{S}_\sigma(u_{n-1})\}$.

Now, according to the below way extend $\sigma$ of $G_0$ to $G$ is edge-coloring $\sigma'$, use colors $A_{\sigma(u)}$ of the color set $\overrightarrow{S}_\sigma(u_i)$ to color edge-coloring $u_i u_i$, where $i = 1, 2, \cdots, n - 1$.

Since $\sigma$ satisfy condition (A), for any adjacent vertex $u_i, u_j \in V(G_0)$ of $G_0$, we have $\overrightarrow{S}_\sigma(u_i) -\{A_{\sigma(u_i)}\} \neq \overrightarrow{S}_\sigma(u_j) -\{A_{\sigma(u_j)}\}$. Therefore, $\overrightarrow{S}_\sigma(u_i) \neq \overrightarrow{S}_\sigma(u_j)$, that is the adjacent vertices $\sigma'$ of $V(G_0)$ is distinguishable. Thanks to $|\overrightarrow{S}_\sigma(u_i)| \geq 2$, then $|\overrightarrow{S}_\sigma(u_i)| \geq 1$, where $i = 1, 2, \cdots, n$. The vertex $u_0$ are different from each adjacent vertex $u_i$ of $G$ since $|\overrightarrow{S}_\sigma(u_0)| \geq 1$. Therefore, $\sigma'$ is $(n-1)$-adjacent vertex distinguishing edge-coloring of $G$. However, $\chi'_d(G) \geq \Delta(G) = n - 1$, thus $\chi'_d(G) = n - 1$.

In this paper, need to use the following Lemma.

Lemma 1.3 (Bazgan et al.[10]) For no isolated edges and no isolated vertex at most one of $G$ for order $n$, we have $\chi'_{vd}(G) \leq n + 1$, where $\chi'_{vd}(G)$ denoted vertex distinguishing edge coloring of graph.
Lemma 1.4 (Zhang et al.[2]) Let $K_m$ be a complete graph of order $m \geq 3$, when $m$ is odd, then $\chi_a'(K_m) = m$; otherwise $\chi_a'(K_m) = m + 1$.

Lemma 1.5 (Zhang et al.[2]) For tree $T$ of order at least 3, if $T$ has adjacent vertices of maximum degree, then $\chi_a'(T) = \Delta(T) + 1$, otherwise $\chi_a'(T) = \Delta(T)$.

In a word, Graph coloring is a sort of models which can be applied in reality such as computer networks, combinatorial optimization, wireless communications, etc. Specifically, for more results about the computer networks and communication technology, the readers may refer to [8-9].

Graph Class $\Omega_n$ Adjacent Vertex Distinguishing Edge Coloring of Lexicographic Product Graphs

Let $G \in \Omega_n$ and $H$ are simple graphs for the vertex set $V(G) = \{u_0, u_1, \cdots, u_{n-1}\}$, $V(H) = \{v_1, v_2, \cdots, v_m\}$, separately, where $u_0$ is the maximum degree vertex, $n \geq 3$, $m \geq 3$. In $G[H]$, assumed that $x_g = (u_i, v_j)$, and $H_i$ is a copy of $H$ for the vertex $u_i$ in $G$, denote by $X_i = V(H_i) = \{x_{i1}, x_{i2}, \cdots, x_{im} | i = 0, 1, \cdots, n - 1\}$. And $G_{i\ell}$ denotes a $m$-regular complete bipartite graphs with bipartition $(X_k, X_i)$. Note that $X_k$ and $X_i$ are independent set of $G_{i\ell}$.

By definition of lexicographic product, $G[H]$ can be decomposed into three disjoint subgraphs $G^{(1)}$, $G^{(2)}$ and $G^{(3)}$, that is $G[H] = G^{(1)} \cup G^{(2)} \cup G^{(3)}$ (a), where $G^{(1)} = \bigcup_{i=0}^{n-1} H_i$, $G^{(2)} = G_{01}$, $G^{(3)}$ is a induced subgraph of $E(G[H]) - E(G^{(1)} \cup G^{(2)})$ for $G[H]$.

The adjacent vertex distinguishing edge coloring of lexicographic product $G[H]$ has the following results when $G \in \Omega_n$.

Theorem 2.1 If $G \in \Omega_n$ and $H$ is a simple connected graph of order $m \geq 3$, then $\chi'_a(G[H]) \leq (n-1)m + \min\{\chi'(H) + 1, \chi'_a(H)\}$.

Proof. Let $G' = G[H]$. Obviously, we proof $\chi'_a(G[H]) \leq (n-1)m + \chi'(H) + 1$ and $\chi'_a(G[H]) \leq (n-1)m + \chi'_a(H)$.

Since $G \in \Omega_n$, we can use $n - 1$ colors $A_1, A_2, \cdots, A_{n-1}$ to coloring $G$ for adjacent vertex distinguishing edge-coloring. With a similar proof as for the lemma 1.2, and assumed that $\sigma_0(u_0u_i) = A_i$.

We construct $((n-1)m + \chi'(H) + 1)$-adjacent vertex distinguishing edge coloring of $G'$ in order to prove $\chi'_a(G[H]) \leq (n-1)m + \chi'(H) + 1$, let color set as $C = \bigcup_{i=0}^{n-1} A_i$ and $|C| = (n-1)m + \chi'(H) + 1$, where $A_i = \{a_i, a_{i1}, \cdots, a_{im} | i = 1, \cdots, n-1\}$, $A_0 = \{\emptyset\}$, $A_0 = \{C_1, C_2, \cdots, C_{\chi'(H)}\}$. By formula (a), we can construct the $((n-1)m + \chi'_a(H))$-adjacent vertex distinguishing edge-coloring $\sigma$ of $G'$ in three steps:

First, we colors the edge of $G^{(1)}$. The proper edge-coloring of $H_{01}$ use the colors $\chi'(H)$ in $A_0$, for any vertex $u_i$ of $G - \{u_0\}$, and $A_{qi} \in \overline{\sigma_0}(u_i)$ since $|\overline{\sigma_0}(u_i)| \geq 1$. Furthermore, $A_0 \cap A_i = \phi$ since $i = 1, 2, \cdots, n - 1$. By Lemma 1.3, can be used $m + 1$ colors of $A_{qi} \cup \{c_1\}$ to coloring $H_i$ about vertex distinguishing edge coloring.

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Secondly, we colors the edge of \( G^{(2)} \). Because \( \sigma\alpha(u_iu_j) = A_i \), the \( m \)-regular complete bipartite graph \( G_{0i} \) can be uniformly proper edge-coloring with \( A_i \cup A_n \) colors in \( m + 1 \), and the color \( b \) can be colors the \( m \) edge of \( G_{0i} \), and each color of \( A_i \) colors \( m − 1 \) edge of \( G_{0i} \).

Finally, we colors the edge of \( G^{(3)} \) is carried out. In \( G' \), the subgraph \( G_0 \) is colored for each edges \( u_iu_j \) of color \( A_p \) in \( G - \{ u_iu_j \} \), the proper edge-coloring is used the colors \( m \) of \( A_p = \{ a_{p1}, a_{p2}, \cdots, a_{pm} \mid p = 1, 2, \cdots, n − 1 \} \).

Obviously, the above coloring \( \sigma \) is the proper edge-coloring of \( G' \), and any two adjacent vertices in each \( X_i \) are distinguishable. For any two vertex \( x_0r, x_0s \) in \( X_0 \), since \( |S_{\sigma}(x_0r) \cap A_{i1}| = 1 \) and \( |S_{\sigma}(x_0s) \cap A_{i1} | \neq |S_{\sigma}(x_0s) \cap A_{i1} | \), \( G' \) exists \( A_{y} \in \{ A_{11}, A_{12}, A_{n-1} \} \) such that \( A_{y} \in S_{\sigma}(u_i) \), \( A_{y} \in S_{\sigma}(u_i) \). Therefore, for any \( x_i \in X_i \), \( x_j \in X_j \) in the under \( \sigma \), we have \( |S_{\sigma}(x_{i1}) \cap A_{y} | = m \), \( |S_{\sigma}(x_{i2}) \cap A_{y} | = m - 1 \). Thus, \( |S_{\sigma}(x_{ij}) \cap A_{y} | = m - 1 \).

In summary, \( \sigma \) is \( G' \) adjacent vertex distinguishing edge-coloring, consequently, \( \chi'_{a}(G[H]) \leq (n-1)m + \chi'(H) + 1 \).

It has been proved that \( \chi'_{a}(G[H]) \leq (n-1)m + \chi'(H) \). With a similar proof as for the above.

**Theorem 2.2** If \( G \in \Omega_n \) and \( H \) is \( m \geq 2 \) complement graphs of order complete graphs, then \( \chi'_{a}(G[H]) = (n-1)m \).

**Proof.** Assumed that \( G' = G[H] \). Obviously, \( \chi'_{a}(G') \geq \Delta(G') = (n-1)m \). It has been proved that \( \chi'_{a}(G') \leq (n-1)m \), With a similar proof as for the Theorem 2.1.

From Lemma 1.1 and Theorem 2.1, the following corollary can be obtained directly.

**Corollary 2.1** Let \( G \in \Omega_n \), \( H \) is a connected graph of order \( m \) with adjacent vertices of maximum degree. If \( \chi'(H) = \Delta(H) \), then \( \chi'_{a}(G[H]) = (n-1)m + \Delta(H) + 1 \).

When \( m \) is odd, then \( \chi'(K_m) = m \); when \( m \) is even, then \( \chi'(K_m) = m - 1 \). By Lemma 1.4 and Theorem 2.1, the following corollary can be obtained directly.

**Corollary 2.2** If \( G \in \Omega_n \), \( H \) is complete graph \( K_m \) of order \( m \geq 3 \), then \( \chi'_{a}(G[H]) = nm \).

By Lemma 1.1, Lemma 1.5 and Theorem 2.1, the following corollary can be obtained directly.

**Corollary 2.3** Assumed that \( G \in \Omega_n \), \( H \) is \( m \geq 3 \) order tree. If \( H \) exists adjacent vertices of the maximum degree, then \( \chi'_{a}(G[H]) = (n-1)m + \Delta(H) + 1 \); Otherwise \( \chi'_{a}(G[H]) = (n-1)m + \Delta(H) \).

According to Lemma 1.1 and Theorem 2.1 the following corollary can be obtained directly.
Corollary 2.4 If $H$ is bipartite graph of order $m \geq 3$, then $\chi_a'(G[H]) = (n-1)m + 3$.

From the definition of lexicographic product, for two graphs $G$ and $H$, equation $G[H] = H[G]$ is generally not set up. Is equation $\chi_a'(G[H]) = \chi_a'(H[G])$ hold? The answer is no, let $W_8$ of order 8 and circle of order 6, we have $\chi_a'(W_8[C_6]) \neq \chi_a'(C_6[W_8])$. In fact, $\chi_a'(W_8[C_6]) = (8-1)6 + 3 = 45$. It is easy to verify that $C_6[W_8]$ exists 30-adjacent vertex distinguishing edge-coloring, therefore, $\chi_a'(W_8[C_6]) \neq \chi_a'(C_6[W_8])$.

What situations will satisfy for graph $G$ and $H$, we have the following Theorem:

Theorem 2.3 If $H \in \Omega_m, G \in \Omega_n$, then $\chi_a'(G[H]) = \chi_a'(H[G]) = mn - 1$.

Proof. By Lemma 1.2 and Theorem 2.1, we have $\chi_a'(H) = \Delta(H) = m - 1$ and $\chi_a'(G[H]) \leq (n-1)m + m - 1 = mn - 1$ since $H \in \Omega_m, G \in \Omega_n$. Where $\chi_a'(G[H]) \geq \Delta(G[H]) = mn - 1$. Hence, $\chi_a'(G[H]) = mn - 1$. Similarly, we can prove that $\chi_a'(H[G]) = mn - 1$. Therefore, $\chi_a'(G[H]) = \chi_a'(H[G]) = mn - 1$.

Let $G_1, G_2, \ldots, G_k$ be $k$ order simple connected graph. We used $G_i[G_{i-1}[\cdots G_3[G_2[G_1]]\cdots]]$ denotes lexicographic product of graph $G_i$ and graph $G_{i-1}[\cdots G_3[G_2[G_1]]\cdots], i = 2, 3, \ldots, k$. If $G_1, G_2, \ldots, G_k$ is the member of $\Omega_n$, then adjacent vertex distinguishing edge coloring of $G_i[G_{i-1}[\cdots G_3[G_2[G_1]]\cdots]]$, have the following conclusions:

Theorem 2.4 For any $k$ 'th element $G_1, G_2, \ldots, G_k$ of $\Omega_n$, where $k \geq 2$, we have $\chi_a'(G_k[G_{k-1}[\cdots G_3[G_2[G_1]]\cdots]]) = n^{k-1} - 1$.

Proof. It is easy to prove that the order and the maximum degree of graph $G_i[G_{i-1}[\cdots G_3[G_2[G_1]]\cdots]]$ is $n^i$, $n^i - 1$, separately, where $i = 2, 3, \ldots, k$. Thus, $\chi_a'(G_k[G_{k-1}[\cdots G_3[G_2[G_1]]\cdots]]) \geq (n-1)n^{k-1} + n^{k-1} - 1 = n^k - 1$.

By mathematical induction, we prove $\chi_a'(G_i[G_{i-1}[\cdots G_3[G_2[G_1]]\cdots]]) \leq n^k - 1$.

By Theorem 2.3, $\chi_a'(G_2[G_1]) = n^2 - 1$ when $k = 2$. If for any $p < k$, Theorem conclusion is established.

Among $G_k[G_{k-1}[\cdots G_3[G_2[G_1]]\cdots]]$, from the inductive hypothesis,

$\chi_a'(G_{k-1}[G_{k-2}[\cdots G_3[G_2[G_1]]\cdots]]) = n^{k-1} - 1$.

By Theorem 2.1, according to $G_k \in \Omega_n$, we have

$\chi_a'(G_k[G_{k-1}[\cdots G_3[G_2[G_1]]\cdots]]) \leq (n-1)n^{k-1} + n^{k-1} - 1 = n^k - 1$.

Hence, $\chi_a'(G_k[G_{k-1}[\cdots G_3[G_2[G_1]]\cdots]]) = n^k - 1$.

Summary

For the adjacent vertex distinguishing edge coloring of lexicographic product, we have the following results. According to the results of Theorem 2.1-2.3, if $H$ is complement graphs of order $m \geq 2$, then $\chi_a'(G[H]) = (n-1)m$; if $H$ is a simple connected graph of order $m \geq 3$, then $\chi_a'(G[H]) \leq (n-1)m + \min(\chi(H) + 1, \chi_a'(H))$; if $H$ is special graphs $\Omega_m$, then $\chi_a'(G[H]) = \chi_a'(H[G]) = mn - 1$. And by Theorem 2.4, for $k \geq 2$, we have $\chi_a'(G_k[G_{k-1}[\cdots G_3[G_2[G_1]]\cdots]]) = n^k - 1$. 

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