Boundedness of Fractional Integral Operators in Weighted Weak Hardy Spaces on Homogeneous Spaces

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Abstract. In this paper, we shall study the theory of weighted weak Hardy spaces $H^{p,\infty}_\omega$ on space of homogeneous type satisfying some reverse doubling condition. More precisely, we will give atomic decomposition characterizations of $H^{p,\infty}_\omega$. Then we use this decomposition to derive the boundedness of fractional integral operators in $H^{p,\infty}_\omega$ and prove an $H^{p,\infty}_\omega$ interpolation theorem. As an applications, the boundedness of Nagel-Stein’s singular and fractional integral operators in $H^{p,\infty}_\omega$ are derived.

Introduction

Weak Hardy Space theory has a very important position in harmonic analysis, Because it sharpens many weak estimates of important operator endpoints. The weak Hardy space was first studied as a special Hardy-Lorentz space [1]. Fefferman and Soria [2] give the atomic decomposition characteristics of the weak Hardy space $H^{1,\infty}$ in the literature. In 1991, Liu Heping [3] established the weak Hardy space $H^{p,\infty}$ theory on homogeneous groups. Combined with the knowledge of homogeneous spatial analysis and analysis theory, in 2009, Ding Yong and Wu Xinfeng [4] established the weak Hardy space $H^{1,\infty}$ theory on homogeneous space, in that paper, the boundedness of $H^{1,\infty}$ to $L^{1,\infty}$ for singular integral operators and fractional integral operators. As an application, the theory can be applied to different backgrounds, such as the European space with $A_\infty$ weight, the Lie group with polynomial volume growth and the Ahlfors n regular metric measure space. Later, Wu [5] extended it to the weak Hardy space $H^{p,\infty}$ of the homogeneous space, and established the atomic decomposition characteristics of $H^{p,\infty}$. The atomic decomposition feature was used to prove the fractional integral operator in The boundedness on $H^{p,\infty}$ and the $H^{p,\infty}$ interpolation theorem. We know that weighted weak Hardy spaces are a good alternative to weighted Hardy spaces when studying operator bounded. Yang [6] established the boundedness theory of operators on weighted weak Hardy spaces on $\mathbb{R}^n$. The boundedness of related operators on weighted Lebesgue spaces, weighted weak Lebesgue spaces, weighted weak Hardy spaces and weighted weak Hardy spaces is discussed. In 1994, Liu Wei [7] established the molecular characterization theory of weighted Hardy spaces on homogeneous spaces.

Now, we study the object of the weighted weak Hardy space $H^{p,\infty}_\omega$ on the homogeneous space. The first, try to establish the atomic decomposition features of $H^{p,\infty}_\omega$, and use this decomposition feature to prove the boundedness of fractional integral operators on $H^{p,\infty}_\omega$ and establish $H^{p,\infty}_\omega$ interpolation theorem. As an application, we obtain the boundedness of the Nagel-Stein singular integral operator and the corresponding fractional integral operator on $H^{p,\infty}_\omega$. The results we get can be applied to more different backgrounds in addition to the above examples.

Definition

Definition 1: Homogeneous Space

Let $(X, d)$ be a metric space with a regular Borel measure $\mu$. All of the metric balls $B(x, r) = \{ y \in X : d(x, y) < r \}$ have a finite positive measure, and for any $x \in X$ and $r > 0$, the quasi-metric satisfies the pseudo-triangle inequality.
\[ d(x, z) \leq \tau(d(x, y) + d(y, z)) \]  
\((X, d, \mu)\) is homogeneous space, if there is still a constant \(C_1 \geq 1\), so that for all \(x \in X\) and \(r > 0\),

\[ \mu(B(x, 2r)) \leq C_1 \mu(B(x, r)) \]  
Before giving the definition of the weighted weak Hardy space \(H_{o}^{P, \infty}(X)\), recall some basic concepts [4,7].

Let \(\varepsilon_1 \in (0,1], \varepsilon_2 > 0\) and \(\varepsilon_3 > 0\). The sequence \(\{S_k\}_{k \in \mathbb{Z}}\) of the bounded linear integral operator on \(L^2(X)\) is called \((\varepsilon_1, \varepsilon_2, \varepsilon_3)\) Identical approximation. If there is a constant \(C_4 > 0\), the integral kernel \(k \in \mathbb{Z}\) for all \(k \in \mathbb{Z}\), all \(x, x', y, y' \in X\) and \(S_k\) is the function from \(X \times X\) to \(C\). Satisfied that:

1. \(|S_k(x, y)| \leq C_4 \frac{1}{V_{2-k}(x) + V_{2-k}(y) + V(x, y)} \times \frac{2^{-k\varepsilon_2}}{(2^{-k} + d(x, y))^{\varepsilon_2}};\)
2. when \(d(x, x') \leq (2^{-k} + d(x, y))/2\),

\[ |S_k(x, y) - S_k(x', y)| \leq C_4 \frac{d(x, x')^{\varepsilon_1} + d(y, y')^{\varepsilon_2}}{(2^{-k} + d(x, y))^{\varepsilon_1} + V_{2-k}(x) + V_{2-k}(y) + V(x, y)} \times \frac{2^{-k\varepsilon_2}}{(2^{-k} + d(x, y))^{\varepsilon_2}};\]
3. \((x, y)\) interchange, nature (ii) established;
4. when \(d(x, x') \leq (2^{-k} + d(x, y))/3\) and \(d(y, y') \leq (2^{-k} + d(x, y))/3\),

\[ |[S_k(x, y) - S_k(x', y')]| - |S_k(x, y) - S_k(x', y')| \leq C_4 \frac{d(x, x')^{\varepsilon_1} + d(y, y')^{\varepsilon_2}}{(2^{-k} + d(x, y))^{\varepsilon_1} + V_{2-k}(x) + V_{2-k}(y) + V(x, y)} \times \frac{2^{-k\varepsilon_2}}{(2^{-k} + d(x, y))^{\varepsilon_2}};\]
5. \(\int_{X} S_k(x, y) d\mu(y) = \int_{X} S_k(x, y) d\mu(x) = 1\).

Let \(x \in X, r \in (0, \infty), \beta \in (0,1]\) and \(\gamma \in (0, \infty)\). The function \(\varphi\) on \(X\) is called the experimental function of the \((x, r, \beta, \gamma)\) type. In case

1. For all \(x \in X\), \(|\varphi(x)| \leq C \frac{1}{\mu(B(xr+d(x,x)) \times (\frac{r}{r+d(x,x)})^\gamma;\)
2. For all \(x, y \in X\) such that \(d(x, y) \leq (r + d(x, x))/2\),

\[ |\varphi(x) - \varphi(y)| \leq C \frac{d(x, y)}{(r + d(x, x))} \times \frac{1}{\mu(B(xr+d(x,x)) \times (\frac{r}{r+d(x,x)})^\gamma.\]

Let \(G(x, r, \beta, \gamma)\) be the set of all test functions of type \((x, r, \beta, \gamma)\). If \(\varphi \in G(x, r, \beta, \gamma)\), define its norm as \(\|\varphi\|_{G(x, r, \beta, \gamma)} = \inf[C: (i)\) and (ii)\) Integrity\). The space \(G(x, r, \beta, \gamma)\) is called the test function space.

In the above definition, fixed \(x_1 \in X\). Let \(G(\beta, \gamma) = G(x_1, 1, \beta, \gamma)\). It is easy to find that for any \(x_2 \in X\) and \(r > 0\), \(G(x_2, r, \beta, \gamma)\) has an equivalent norm. When \(\gamma \in (0, \varepsilon]\), any given \(\varepsilon \in (0,1]\), \(G_0(\beta, \gamma)\) be the completion space in \(G(\beta, \gamma)\). \(\varphi \in G_0(\beta, \gamma)\) if and only if \(\varphi \in G(\beta, \gamma)\) and there exists \(\{\varphi_i\}_{i \in \mathbb{N}}\) such that \(\|\varphi - \varphi_i\|_{G(\beta, \gamma)} \rightarrow 0\), when \(i \rightarrow \infty\). If \(\varphi \in G_0(\beta, \gamma)\), define \(\|\varphi\|_{G_0(\beta, \gamma)} = \|\varphi\|_{G(\beta, \gamma)}\). Obviously, \(G_0(\beta, \gamma)\) is the Banach space. \(G_0(\beta, \gamma)'\) represents the dual space of \(G_0(\beta, \gamma)\).

Let \(\varepsilon_1 \in (0,1]\), \(\varepsilon_2 > 0\), \(\varepsilon_3 > 0\), \(0 < \varepsilon < \min(\varepsilon_1, \varepsilon_2)\), \(\{S_k\}_{k \in \mathbb{Z}}\) is \((\varepsilon_1, \varepsilon_2, \varepsilon_3)\) - Constant approximation. For \(f \in (G_0(\beta, \gamma)')\) and \(\beta, \gamma \in (0, \varepsilon)\), the non-tangential maximal operator \(M_\varepsilon\) is defined as:

\[ M_\varepsilon(f)(x) := \sup_{k \in \mathbb{Z}, d(x, y) \leq \varepsilon 2^{-k}} |S_k(f)(y)|.\]
Grand Maximal Operator $M_g$ is defined as [6]:

$$M_g f(x) := \sup \{|\langle f, \varphi \rangle|: \varphi \in G_0^\epsilon(\beta, \gamma), \| \varphi \|_{G(x, r, \beta, \gamma)} \leq 1 \text{ for some } r > 0\}. $$

The Radial Maximal Opertor $M_0$ is defined as [6]:

$$M_0 f(x) := \sup_{k \in \mathbb{Z}} |S_k(f)(x)|. $$

Definition 2: Weighted weak Hardy space $H^p,\infty(\mathcal{X})$ [15,16]

Let $\epsilon_1 \in (0,1], \epsilon_2 > 0, \epsilon_3 > 0, \epsilon \in (0, \min\{\epsilon_1, \epsilon_2\}), \{S_k\}_{k \in \mathbb{Z}}$ is $(\epsilon_1, \epsilon_2, \epsilon_3)$ - Constant approximation. When $\beta, \gamma \in (0, \epsilon)$ let $p \in (D/(D + 1),1], \sigma \in (0, \infty)$ and $f \in (G_0^\epsilon(\beta, \gamma))'$. Weighted weak Hardy space $H^p,\infty_\omega(\mathcal{X})$ on the homogeneous space is defined as

$$H^p,\infty_\omega(\mathcal{X}) = \{f \in (G_0^\epsilon(\beta, \gamma))': M_\sigma f \in L^p,\infty_\omega(\mathcal{X})\}. $$

The $H^p,\infty_\omega$ quasi-norm of $f$ is defined as:

$$\| f \|_{H^p,\infty_\omega(\mathcal{X})} = \| M_\sigma f \|_{L^p,\infty_\omega(\mathcal{X})}. $$

Theorem and Proof

Theorem 1: Let $p \in (D/(D + 1),1].$ Given $f \in H^p,\infty_\omega(\mathcal{X}),$ there exists a bounded function sequence $\{f_k\}_{k=-\infty}^{\infty},$ which satisfies the following conditions:

1. $f - \sum_{k \leq N} f_k \to 0$ convergence in $(G_0^\epsilon(\beta, \gamma))'.$

2. Each $f_k$ can be decomposed into $f_k = \sum_{i=1}^{\infty} h_i^k$ in the sense of $(G_0^\epsilon(\beta, \gamma))'$, and $h_i^k$ such that:
   a. The $h_i^k$ support is on $B_i^k$, for each $k$, $\{B_i^k\}$ only has a finite overlap;
   b. $\int_{B_i^k} h_i^k = 0$;
   c. $\| h_i^k \|_{L^\infty} \leq C2^k$ and $\sum_i \omega(B_i^k) \leq C_{12}^{-kp}$, constant $C_1 \approx \| f \|_{H^p,\infty_\omega(\mathcal{X})}.$

Conversely, if $f$ is a distribution satisfying (a) and (b) (i)-(iii), then $f \in H^p,\infty_\omega(\mathcal{X})$ and $\| f \|_{H^p,\infty_\omega(\mathcal{X})} \leq c C_1$, where $c$ is an absolute constant.

Proof:

For $x \in Z$, Let $\Omega_k = \{x \in \mathcal{X}: M_g f(x) > 2^k\}$. Then for any $x \in Z$, $\Omega_k$ is the open subset of $\mathcal{X}$, which satisfying $\omega(\Omega_k) \leq C2^{-kp} \| f \|_{H^p,\infty_\omega(\mathcal{X})} < \infty$. From the conclusions in the lemmas 2 and lemma 3, let $\{B_i^k\}_{i=1}^{\infty} = \{B(x_i^k, r_i^k)\}_{i=1}^{\infty}$ is the Whitney decomposition of $\Omega_k$, let $\varphi_i^k$ and $B_i^k$ be related to the “bump function”. For each $x \in Z$, use inf $\{|d(x,y)|: y \notin \Omega_k\}$ express. definition

$$m_i^k = \frac{1}{\int_X \varphi_i^k} \int_X f \varphi_i^k. $$

Decompose $f$ into

$$f(x) = (f(x) \chi_{\Omega_k}(x) + \sum_{i=1}^{\infty} m_i^k \varphi_i^k(x)) + \sum_{i=1}^{\infty} (f(x) - m_i^k) \varphi_i^k(x). $$

let

$$g_k(x) := (f(x) \chi_{\Omega_k}(x) + \sum_{i=1}^{\infty} m_i^k \varphi_i^k(x)).$$

Easy to proof
\[ |f(x)\chi_{\alpha_k}(x)| \leq C(M_g f)(x)\chi_{\alpha_k}(x) \leq C2^k \tag{3} \]

Thus, for all \( \in X \), \( |g_k(x)| \leq C2^k \). can be uniformly converged.

\[
\lim_{k \to -\infty} g_k(x) = 0. \tag{4}
\]

On the other hand, note that \( |\Omega_k| = O(2^{-kp}) \to 0 \), when \( k \to \infty \),

\[
\lim g_k(x) = f(x), a.e \tag{5}
\]

By (4) and (5), get

\[
f = \sum_{k=-\infty}^{\infty} g_{k+1} - g_k := \sum_{k=-\infty}^{\infty} f_k, \ a.e
\]

So

\[
f_k = \sum_{i=1}^{\infty} \left[ (f - m_i^k)\varphi_i^k - \sum_{j=1}^{\infty} (f - m_{ij}^{k+1})\varphi_i^k \varphi_j^{k+1} \right] + \sum_{j=1}^{\infty} \left[ \sum_{i=1}^{\infty} (f - m_{ij}^{k+1})\varphi_i^k \varphi_j^{k+1} - (f - m_j^{k+1})\varphi_j^{k+1} \right], \tag{6}
\]

Where all sequences converge in the sense of \((G^{(k)}(\beta, \gamma))'\) and

\[ m_{ij}^{k+1} = \frac{1}{\int \varphi_i^k \varphi_j^{k+1} f \varphi_i^k \varphi_j^{k+1}}. \]

Let \( \beta_i^k = (f - m_i^k)\varphi_i^k - \sum_{j=1}^{\infty} (f - m_{ij}^{k+1})\varphi_i^k \varphi_j^{k+1} \) and

\[ y_j^{k+1} = \sum_{i=1}^{\infty} (f - m_{ij}^{k+1})\varphi_i^k \varphi_j^{k+1} - (f - m_j^{k+1})\varphi_j^{k+1}. \]

Then \( \text{supp} \beta_i^k \subset B(x_i^k, 2r_i^k) \) and \( \text{supp} y_j^{k+1} \subset B(x_j^{k+1}, 2r_{j+1}^k) \).

Obviously

\[ B(x_j^{k+1}, 2r_{j+1}^k) \subset \Omega_{k+1} \subset B(x_i^k, r_i^k), \]

Then exists \( B(x_i^k, r_i^k) \cap B(x_j^{k+1}, 2r_{j+1}^k) \neq \Phi \)

Let \( \tilde{B}_i^k := B(x_i^k, 7\tau r_i^k) \), \( \tau \) is a constant appears in the triangle infinitive . Then \( \{\tilde{B}_i^k\}_{i=1}^{\infty} \) has bounded overlap, and

\[ B(x_j^{k+1}, 2r_{j+1}^k) \subset \tilde{B}_i^k \]

In fact, for any \( x \in B(x_j^{k+1}, 2r_{j+1}^k) \) and any \( y \in B(x_i^k, r_i^k) \cap B(x_j^{k+1}, 2r_{j+1}^k) \) . Obviously

\[ y \in B(x_i^k, 15r_i^k) \cap B(x_j^{k+1}, 15r_{j+1}^k). \]

Then by lemma 2 (iv)

\[ d(x, x_i^k) \leq \tau[d(x, x_j^{k+1}) + d(x_j^{k+1}, x_i^k)] \leq \tau[4r_{j+1}^k + r_i^k] \leq \tau[(4/15)d(x, y) + r_i^k] \leq 13\tau r_i^k, \tag{8} \]

Including (7). Let \( \tilde{y}_i^k = y_i^k \), so \( \text{supp} \tilde{y}_i^k \subset \tilde{B}_i^k \), then

\[ \text{supp} \beta_i^k \subset \tilde{B}_i^k, \text{ supp} \tilde{y}_i^k \subset \tilde{B}_i^k. \]

Next, defined by (3) bounded overlap \( \{\tilde{B}_i^{k+1}\}_{j=1}^{\infty} \),there is

\[ |\beta_i^k| = |(f - m_i^k)\varphi_i^k - \sum_{j=1}^{\infty} (f - m_{ij}^{k+1})\varphi_i^k \varphi_j^{k+1}| \leq |f\varphi_i^k \chi_{\alpha_k+1}| + |m_i^k\varphi_i^k| + \sum_{j=1}^{\infty} |m_{ij}^{k+1}|\varphi_i^k \varphi_j^{k+1} \leq C2^k. \tag{9} \]

Similarly \( |\tilde{y}_i^k| \leq C2^k \).
\[ \int_{\mathcal{X}} \beta_i^k (x) \, dx = 0 = \int_{\mathcal{X}} y_i^k (x) \, dx. \]

Defined \( h^k_i = \beta^k_i + \gamma^k_i \), then \( f_k = \sum_{i=1}^{\infty} h^k_i \) and the convergence in \( (\mathcal{G}_0(\beta, \gamma))' \) can be used to prove. Thus (i) in the (a) and (ii) in the (b) for theorem 3 have been proved to be valid. Finally, because \( f \in H_{\omega}^{p,\infty} \) and \( \{B^k_i\} \) has bounded overlap, by (2)

\[ \sum_{i=1}^{\infty} \omega(B^k_i) \leq \sum_{i=1}^{\infty} \omega(B^k_i) \leq \omega(\Omega_k) \leq 2^{-kp} \|f\|_{H_{\omega}^{p,\infty}(\mathcal{X})}. \]

This get (iii) in (b). The completion of the atomic decomposition is thus completed.

Fix \( \alpha > 0 \), select \( k_0 \), so that \( 2^{k_0} \leq \alpha < 2^{k_0+1} \). let

\[ f = \sum_{k=-\infty}^{k_0-1} f_k + \sum_{k=k_0}^{\infty} f_k = F_1 + F_2. \]

Because

\[ \mathcal{M}_0(F_1)(x) \leq \sum_{k=-\infty}^{k_0-1} \mathcal{M}_0(f_k)(x) \leq C \sum_{k=-\infty}^{k_0-1} 2^k \leq C_2 \alpha, \]

And \( \omega(\{x \in \mathcal{X} : \mathcal{M}_0(F_1)(x) > C_2 \alpha\}) = 0 \), since

\[ \omega(\{x \in \mathcal{X} : \mathcal{M}_0(f)(x) > (C_2 + 1)\alpha\}) \leq \omega(\{x \in \mathcal{X} : \mathcal{M}_0(F_2)(x) > \alpha\}). \]

let

\[ A_{k_0} = \bigcup_{k=k_0}^{\infty} 3\tau B^k_i, \]

where \( 3\tau B^k_i \) is a sphere for the heart of \( x^k_i \) whose 3 times larger radius. By (2)

\[ \omega(A_{k_0}) \leq (3\tau)^{D} C_{02}^{-k_0p} \leq C/\alpha^p. \]

So just to proof

\[ I = \omega(\{x \in A_{k_0} : \mathcal{M}_0(F_2)(x) > \alpha\}) \leq \frac{C}{\alpha^p} \]

Note that for \( x \notin 3\tau B^k_i \) and \( y \in B^k_i \), there is

\[ d(x,y) \geq \frac{1}{\tau} d(x,x^k_i) - d(y,x^k_i) \geq 2d(y,x^k_i). \]

since

\[ |S_j(x,y) - S_j(x,x^k_i)| \lesssim \frac{d(y,x^k_i)^{\varepsilon_1}}{d(x,y)^{\varepsilon_1} V(x,y)}. \]

Then by the vanishing condition of \( h^k_i \),

\[ \mathcal{M}_0(h^k_i)(x) = \sup_{f} |\int S_j(x,y) - S_j(x,x^k_i) | h^k_i(y) dy| \]

\[ \leq C 2^k \frac{\mu(B^k_i) d(x,x^k_i)^{\varepsilon_1}}{V(x,y) d(x,y)^{\varepsilon_1}} \]

\[ \leq C 2^k \frac{\mu(B^k_i)(r^k_i)^{\varepsilon_1}}{\mu(B(x^k_i,d(x,x^k_i))) d(x,x^k_i)^{\varepsilon_1}}. \]
by (1.4)(1.5)

\[ \mu(B(x_i^k, d(x, x_i^k))) \leq \left( \frac{d(x, x_i^k)}{r_i^k} \right)^D \mu(B_i^k). \]

since,

\[ \mathcal{M}_0(h_i^k)(x) \leq C2^k \frac{\mu(B_i^k)}{V(x, x_i^k)} \leq C_{\varepsilon_1, n} 2^k \frac{\omega(B_i^k)}{V'(x, x_i^k)}. \]

Apply the lemma 4 to \( g_{ki} = V'(x, x_i^k)^{-1} \) \( \varepsilon_2, r = (1 + \varepsilon_2)^{-1}, \) and \( c_{ki} = 2^k \omega(B_i^k)^{1 + \varepsilon_2}. \)

\[ I \leq C_{\varepsilon_1, D} \frac{\alpha^r}{\alpha^q} \sum_{k \geq k_0} \sum_i 2^{kr} \omega(B_i^k) \leq C_1 \frac{C_{\varepsilon_1, D}}{\alpha^r} \sum_{k \geq k_0} 2^{kr} 2^{-kp} \]

because \( D/(D + \varepsilon) > r, \) by

\[ C_1 \frac{C_{\varepsilon_1, D}}{\alpha^q} 2^{-k_0(p-r)} = C/\alpha^p, \]

The sequence is convergent and bounded, and the constant \( C \) is independent of \( f \) and \( \alpha. \) This completes the proof of the theorem 1.

As an application of atomic decomposition, we prove the following interpolation theorem, which generalizes the conclusions of references 4[4].

Theorem 2: Let \( D/(D + 1) < q < p \leq 1 < p_0 < \infty. \) Let \( T \) be a sub-additive operator, \( T \) is \((L_{\omega}^{p_0}(X), L_{\omega}^{p_0}(X))\) and \((H_{\omega}^{p_0}(X), H_{\omega}^{p_0}(X))\) type. Then \( T \) is bounded on \( H_{\omega}^{p_0}(X). \)

Proof: for each of \( f \in H_{\omega}^{p_0}(X) \) and \( \lambda > 0, \) proof

\[ \omega(\{x \in X : M_g(TF_1)(x) > \lambda\}) \leq C \lambda^{-p} \| f \| H_{\omega}^{p_0}(X) \]

And constant \( C \) is independent of \( f \) and \( \lambda. \)

Select \( k_0 \in Z, \) such that \( 2^{k_0} \leq \lambda < 2^{k_0+1}. \) By the atomic decomposition of \( H_{\omega}^{p_0}(X), \) write \( f = \sum_{k=-\infty}^{k_0} f_k + \sum_{k=k_0+1}^{\infty} f_k := F_1 + F_2. \) Note \( p_0 > 1, \) there is

\[ \| F_2 \| L_{\omega}^{p_0}(x) \leq C \sum_{k=-\infty}^{k_0} \| f_k \| L_{\omega}^{p_0}(x) \leq C \sum_{k=-\infty}^{k_0} 2^k \left( \sum_i \omega(B_i^k) \right)^{1/p_0} \leq C \| f \|_{H_{\omega}^{p_0}(x)}^{p/p_0} 2^{k_0(1-p/p_0)}. \]

since

\[ \omega(\{x \in X : M_g(TF_1)(x) > \lambda\}) \leq \lambda^{-p_0} \| M_g(TF_1) \|_{L_{\omega}^{p_0}(X)}^{p_0} \]

\[ \leq C \lambda^{-p_0} \| TF_1 \|_{H_{\omega}^{p_0}(X)}^{p_0} \]

\[ \leq C \lambda^{-p_0} \| F_1 \|_{H_{\omega}^{p_0}(X)} \]

\[ \leq C \lambda^{-p_0} \| f \|_{H_{\omega}^{p_0}(X)} 2^{k_0(p_0-p)} \]

\[ \leq C \lambda^{-p_0} \| f \|_{H_{\omega}^{p_0}(X)} \lambda^{p_0-p} \]

\[ = C \lambda^{-p} \| f \|_{H_{\omega}^{p_0}(X)} \]

So, we just need to prove

\[ \omega(\{x \in X : M_g(TF_2)(x) > \lambda\}) \leq C \lambda^{-p} \| f \|_{H_{\omega}^{p_0}(X)} \]

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Notes $\mathcal{M}_g(TF_2)(x) \leq \sum_{k=k_0}^{\infty} \mathcal{M}_N(f_k)(x)$, and $f_k = \sum_i h_i^k$. Easy to catch $C^{-1} 2^{-k} \omega(B_l^k)^{-1/q} h_l^k$ is atom of $(q, \infty)$. since $f_k \in H^{q}_{\omega}(X)$ and

$$
\|f_k\|_{H^{q}_{\omega}(X)} \leq C \sum_i 2^{kq} \omega(B_l^k) \leq C 2^{k(q-p)} \|f\|_{H^{p,\infty}_{\omega}(X)}.
$$

Because $T$ is bounded on $H^{q}_{\omega}(X)$,

$$
\omega(\{x \in X : \mathcal{M}_g(Tf_k)(x) > \lambda\}) \leq C \lambda^{-q} \|Tf_k\|_{H^{q}_{\omega}(X)} \leq C \lambda^{-q} \|f_k\|_{H^{q}_{\omega}(X)}.
$$

since

$$
\omega(\{x \in X : \mathcal{M}_g(T(f_k/\|f_k\|_{H^{q}_{\omega}(X)}))(x) > \lambda\}) \leq C \lambda^{-q}.
$$

By lemma 1:

$$
\omega(\{x \in X : \mathcal{M}_g(TF_2)(x) > \lambda\})
\leq \omega(\{x \in X : \sum_{k=k_0+1}^{\infty} \|f_k\|_{H^{q}_{\omega}(X)} \mathcal{M}_g(T(f_k/\|f_k\|_{H^{q}_{\omega}(X)}))(x) > \lambda\})
\leq \frac{2-q}{1-q} \frac{1}{\lambda^q} \sum_{k=k_0+1}^{\infty} \|f_k\|_{H^{q}_{\omega}(X)}
\leq \frac{C \|f\|_{H^{p,\infty}_{\omega}(X)}}{\lambda^q} \sum_{k=k_0}^{\infty} 2^{k(q-p)}
\leq C 2^{k_0(q-p)} \|f\|_{H^{p,\infty}_{\omega}(X)}/\lambda^q
\leq C \lambda^{-p} \|f\|_{H^{p,\infty}_{\omega}(X)}.
$$

This completes the proof of the theorem 2.

For the fractional integral operator defined below

$$
T_\alpha f(x) = \int_X K_\alpha(x,y) f(y) dy,
$$

get

**Theorem 3:** Let $1 < p_0 < q_0 < \infty$ satisfy $1/p_0 - 1/q_0 = \alpha$ and $0 < \alpha < 1$. The operator $T$ is bounded from $L^{p_0}_{\omega}(X)$ to $L^{q_0}_{\omega}(X)$. If K also satisfies the following regularity condition: i.e there is a constant $C$, $\epsilon > 0$ such that $d(y,y') \leq d(x,y)/2$ for all $x, y, y' \in X$,

$$
|K_\alpha(x,y) - K_\alpha(x,y')| \leq C \frac{d(y,y')^\epsilon}{\nu(x,y)^{1-\epsilon} d(x,y)^\epsilon}, \tag{11}
$$

when $0 < \alpha < \min\{\epsilon, 1\}/n$, for all $p, q$ satisfies $n/(n+1) < p < q \leq 1$ and $1/p - 1/q = \alpha$, $T_\alpha$ is bounded from $H^{p,\infty}_{\omega}(X)$ to $L^{q,\infty}_{\omega}(X)$. That is, there is a constant $C > 0$, which is independence on $f$ and $\lambda$, for all $\lambda > 0$, there is

$$
\omega(\{x : |T_\alpha f(x)| > \lambda\}) \leq C \left( \frac{\|f\|_{H^{p,\infty}_{\omega}(X)}}{\lambda} \right)^q.
$$

**Proof:** Fixed $\lambda$. let $\eta = \lambda^{q/p} \|f\|_{H^{p,\infty}_{\omega}(X)}$. Let $k_0 \in \mathbb{Z}$, get $2^{k_0} \leq \eta < 2^{k_0+1}$. Then decomposition $f$ into two parts
\[ f = \sum_{k=-\infty}^{K_0} f_k + \sum_{k=K_0+1}^{\infty} f_k : = F_3 + F_4. \]

By Atomic decomposition of the function \( f \) on \( H_{\omega, \infty}^p(\mathcal{X}) \)

\[
\| F_3 \|_{L_{\omega}^{p_0}(\mathcal{X})} \leq \sum_{k=-\infty}^{K_0} \| f_k \|_{L_{\omega}^{p_0}(\mathcal{X})}
\leq C \sum_{k=-\infty}^{K_0} 2^k \left( \sum_i \omega \left( B_i^k \right) \right)^{1/p_0}
\leq C \| f \|_{H_{\omega, \infty}^p(\mathcal{X})} \sum_{k=-\infty}^{K_0} 2^{k(1-1/p_0)}
\leq C \| f \|_{H_{\omega, \infty}^p(\mathcal{X})} \eta^{1-1/p_0}
= C \lambda^{q(1/p-1/p_0)} \| f \|_{H_{\omega, \infty}^{p_0}(\mathcal{X})}^{1-q(1/p-1/p_0)}. \tag{12}
\]

By the bounded of \( T_\alpha \) which \( L_{\omega}^{p_0}(\mathcal{X}) \to L_{\omega}^{q_0}(\mathcal{X}) \) and the bounded of (12)

\[ \omega(\{ x \in \mathcal{X} : |T_\alpha F_3(x)| > \lambda \}) \leq c \lambda^{-q_0} \| T_\alpha F_3 \|_{L_{\omega}^{q_0}(\mathcal{X})}^{q_0} \]

\[ \leq c \lambda^{-q_0} \| F_3 \|_{L_{\omega}^{p_0}(\mathcal{X})}^{q_0} \]
\[ \leq c \lambda^{-q_0} \left( \lambda^{q(1/p-1/p_0)} \right) \| f \|_{H_{\omega, \infty}^{p_0}(\mathcal{X})}^{q_0}
= C \left( \frac{\| f \|_{H_{\omega, \infty}^{p_0}(\mathcal{X})}}{\lambda} \right)^q. \tag{13} \]

Let \( \hat{B}_i^k = 3 \tau B_i^k \) and

\[ E_{\kappa_0} = \bigcup_{k=K_0+1}^{\infty} \bigcup_i \hat{B}_i^k. \]

We get

\[ \omega(E_{\kappa_0}) \leq C \sum_{k=K_0+1}^{\infty} \sum_i \omega \left( B_i^k \right)
\leq C \| f \|_{H_{\omega, \infty}^p(\mathcal{X})} \sum_{k=K_0+1}^{\infty} 2^{-kp}
\leq C \| f \|_{H_{\omega, \infty}^p(\mathcal{X})} \eta^{-p} \tag{13} \]

To complete the proof of the theorem, just prove

\[ \omega(\{ x \in E_{\kappa_0}^c : |T_\alpha F_4(x)| > \lambda \}) \leq C \left( \frac{\| f \|_{H_{\omega, \infty}^{p_0}(\mathcal{X})}}{\lambda} \right)^q. \tag{14} \]

Notes, if \( x \in E_{\kappa_0}^c \) and \( y \in B_i^k \), then by (1)

\[ d(x, y) \geq \frac{1}{\tau} d(x, \hat{x}_i^k) - d(x_i^k, y) \geq 2 d(x_i^k, y). \]

Thus by the vanishing condition of \( h_i^k \), Minkowski’s in equality and (11),
\[
\omega(\{x \in E^c_{k_0} : |T_\alpha F_k(x)| > \lambda\}) 
\leq \lambda^{-1} \int_{E^c_{k_0}} |T_\alpha F_k(x)| \omega dx 
\leq \lambda^{-1} \sum_{k=k_0+1}^\infty \sum_{i=0}^\infty \int_{B^k_i} |h^k_i(y)| \int_{E^c_{k_0}} |K(x, y) - K(x, x^k_i)| \omega dx dy 
\leq \lambda^{-1} \sum_{k=k_0+1}^\infty \sum_{i=0}^\infty \int_{B^k_i} |h^k_i(y)| \int_{E^c_{k_0}} \frac{d(y, x^k_i)^\epsilon}{V'(x, y)^{1-\alpha} d(x, y)^\epsilon} \omega dx dy.
\]

By (2)
\[
\int_{E^c_{k_0}} \frac{d(y, x^k_i)^\epsilon}{V'(x, y)^{1-\alpha} d(x, y)^\epsilon} \omega dx = \sum_{j=0}^\infty \int_{2^j d(x^k_i, y) \leq d(x, y) < 2^{j+1} d(x^k_i, y)} \frac{d(y, x^k_i)^\epsilon}{V'(x, y)^{1-\alpha} d(x, y)^\epsilon} \omega dx 
\leq \sum_{j=0}^\infty 2^D 2^{-j \epsilon} \omega(B(y, 2^{j+1} d(x^k_i, y)))^\alpha
\leq C \omega(B^k_i)^\alpha.
\]

Again by (15)
\[
\omega(\{x \in E^c_{k_0} : |T_\alpha F_k(x)| > \lambda\}) \leq C \lambda^{-1} \sum_{k=k_0+1}^\infty \sum_{i=0}^\infty 2^k \left( \sum_{i=0}^\infty \omega(B^k_i) \right)^{1+\alpha}
\leq C \lambda^{-1} \sum_{k=k_0+1}^\infty \sum_{i=0}^\infty 2^k \left( 2^{-k \| f \|_{H^p_{\omega, \infty}}} \right)^{1+\alpha}
= C \lambda^{-1} \| f \|_{H^p_{\omega, \infty}} \left( \lambda \| f \|_{H^p_{\omega, \infty}}^{1+\alpha} \right)^{1+\alpha}
= C \lambda^{-1} \| f \|_{H^p_{\omega, \infty}}^{1+\alpha} \left( \frac{\lambda}{\lambda} \right)^q.
\]

This completes the proof of the theorem 3.

For the boundedness of the fractional integral \((H^p_{\omega, \infty}, H^q_{\omega, \infty})\)

Theorem 4: Under the same assumption as the theorem 3 , If \(T_\alpha\) also satisfies the following vanishing condition: For any \(L^p(\mathcal{X})\) function \(\phi\), which is a compact support set such that \(\int_{\mathcal{X}} \phi = 0\),

\[
\int_{\mathcal{X}} (T_\alpha \phi) (x) dx = 0.
\]

Then \(T_\alpha\) be bounded form \(H^p_{\omega, \infty}(\mathcal{X})\) to \(H^q_{\omega, \infty}(\mathcal{X})\). And there is a constant \(C\) independent of \(f\) and \(\lambda\), such that for each \(\lambda > 0\),

\[
\| f \|_{H^p_{\omega, \infty}} = \| f \|_{H^p_{\omega, \infty}}^{1+\alpha} \left( \frac{\lambda}{\lambda} \right)^q.
\]
\[
\omega(\{x: |(M_0 T_\alpha f)(x)| > \lambda\}) \leq C \left( \frac{\| f \|_{H_\omega^{p,\infty}(X)}^q}{\lambda} \right).
\]

Proof: Because \( f \in H_\omega^{p,\infty}(X), M_0(f) \in L_\omega^{p,\infty}(X) \). To prove the theorem, we only need to prove: for any \( \lambda > 0 \),
\[
\omega(\{x \in X: M_\sigma(T_\alpha f)(x) > \lambda\}) \leq C(\| f \|_{L_\omega^q,\infty}(X)/\lambda)^q.
\]
Assume that \( \| f \|_{H_\omega^{p,\infty}(X)} > 0 \). For each \( \lambda > 0 \), using a similar argument in the proof of (12), we can get
\[
\| F_3 \|_{L_\omega^q(X)} \leq \lambda^{1 - \frac{q}{2}} \| f \|_{H_\omega^{p,\infty}(X)}^q.
\]
By boundedness of \( L_\omega^q(X) \) for \( M_0 \) and boundedness of \( (L_\omega^p, L_\omega^q) \) for \( T_\alpha \), then
\[
\omega(\{x \in X: M_0(T_\alpha F_3)(x) > \lambda\}) \leq C(\| M_0(T_\alpha F_3) \|_{L_\omega^q,\infty}/\lambda)^q \leq C(\| f \|_{L_\omega^{p,\infty}(X)}/\lambda)^q.
\]

Designation \( B_i^k = B(x_i^k, 4r_i^k) \) and \( E = \bigcup_{k=k_0}^{\infty} \bigcup_i B_i^k \). Similar to the proof of \( \omega(E) \leq C(\| f \|_{L_\omega^q,\infty}/\lambda)^q \).

Because \( h_i^k \in L_\omega^0(X) \) has a compact support set, and \( \int_X h_i^k(x)dx = 0 \), by the vanishing condition (16),
\[
\int_X T_\alpha(h_i^k)(x)dx = 0.
\]

Thus, for any \( m \in Z \) and \( x \in E^c \), by \( h_i^k \) and the vanishing condition of (16), there is
\[
|S_m(T_\alpha h_i^k)(x)| = \left| \int_X (T_\alpha h_i^k)(y)[S_m(x,y) - S_m(x,x_i^k)]dy \right| \leq \int_{B_i^k} (T_\alpha h_i^k)(y) \cdot |S_m(x,y) - S_m(x,x_i^k)|dy
\]
\[
+ \int_{2r_i^k \leq d(y,x_i^k)} (\int_{B_i^k} |h_i^k(v)||K_\sigma(y,v) - K_\sigma(y,x_i^k)|dv) \cdot |S_m(x,y) - S_m(x,x_i^k)|dy
\]
\[
+ \int_{d(y,x_i^k) \geq d(x_i^k)} (\int_{B_i^k} |h_i^k(v)||K_\sigma(y,v) - K_\sigma(y,x_i^k)|dv)(|S_m(x,y)| + |S_m(x,x_i^k)|)dy
\]
\[
:= J_1 + J_2 + J_3.
\]
First give the estimation of \( J_1 \). By the definition of \( S_m \),
\[
J_1 \leq \frac{c}{V(x,x_i^k)} \int_{B_i^k} (T_\alpha h_i^k)(y)d(u,x_i^k)\epsilon dy
\]
\[
\leq \frac{c(\epsilon)^{\epsilon}}{V(x,x_i^k)} \| T_\alpha h_i^k \|_{L_\omega^q,\infty}(X) \mu(B_i^k)^{1 - \frac{2}{4}}
\]
\[
\leq C_2 k^{1 + \alpha} \mu(B_i^k)^{1 + \alpha} \frac{(r_i^k)^{\epsilon}}{V(x,x_i^k)}d(x_i^k)^{\epsilon}.
\]
Nest, give the estimation of \( J_2 \). Notes \( d(y,x_i^k) \geq 3\tau r_i^k \) and \( v \in B_i^k \), get \( d(y,v) \geq 1/\tau d(y,x_i^k) - d(v,x_i^k) \geq 2d(v,x_i^k) \). There is,
\[ |K_\alpha(y, v) - K_\alpha(y, x_i^e)| \leq C \frac{d(v, x_i^e)}{\nu(y, x_i^e)^{1-\alpha}d(y, x_i^e)} \quad (21) \]

since

\[ J_2 \leq C_2^k \mu(B_i^k) \int_{2r_i^k < d(y, x_i^e) < d(x, x_i^e)/2} \frac{1}{\nu(y, x_i^e)^{1-\alpha}} \cdot \frac{(r_i^k)^{\epsilon_1}}{\nu(x, x_i^e)^{1-\alpha}} dy \]

\[ \leq C_2^k \mu(B_i^k)(r_i^k)^{\epsilon_1} \cdot \frac{1}{\nu(x, x_i^e)^{1-\alpha}d(x, x_i^e)^{\epsilon_1}}. \]

For \( J_3 \), us (21), again, there is

\[ J_3 \leq C \int_{d(y, x_i^e)d(x, x_i^e)/2} \frac{|h_i^k(v)|d(v, x_i^e)^{\epsilon_1}}{\nu(y, x_i^e)^{1-\alpha}d(y, x_i^e)^{\epsilon_1}} dv \cdot \left( |S_m(x, y)| + \frac{1}{\nu(x, x_i^e)^{1-\alpha}d(y, x_i^e)^{\epsilon_1}} dy \right) \quad (23) \]

\[ \leq \frac{C_2^k \mu(B_i^k)(r_i^k)^{\epsilon_1}}{\nu(x, x_i^e)^{1-\alpha}d(x, x_i^e)^{\epsilon_1}} + \frac{C_2^k \mu(B_i^k)(r_i^k)^{\epsilon_1}}{\nu(x, x_i^e)^{1-\alpha}d(x, x_i^e)^{\epsilon_1}V(x, x_i^e)} \left( \int_{d(y, x_i^e)d(x, x_i^e)/2} \frac{1}{\nu(y, x_i^e)^{1-\alpha}d(y, x_i^e)^{\epsilon_1}} dy \right) \]

Comprehensive (21), (22) and (23) conclusions, For any \( X \in E^c \),

\[ |S_m(T_\alpha h_i^k)(x)| \leq \frac{C_2^k \mu(B_i^k)(r_i^k)^{\epsilon_1}\omega(B(x, d(x, x_i^e))^{\epsilon_1})}{V(x, x_i^e)^{1-\alpha}} \leq \frac{C_2^k \mu(B_i^k)^{1+\epsilon_1}\omega(B_i^k)^{1+\epsilon_1}}{V(x, x_i^e)^{1+\epsilon_1}}. \]

For any \( X \in E^c \),

\[ M_0(T_\alpha h_i^k)(x) \leq \frac{C_2^k \mu(B_i^k)^{1+\epsilon_1}\omega(B_i^k)^{1+\epsilon_1}}{V(x, x_i^e)^{1+\epsilon_1}}. \]

Let \( C_i^k = C_2^k \omega(B_i^k)^{1+\epsilon_1} \) and \( g_i^k(x) = \frac{\mu(B_i^k)^{1+\epsilon_1}}{V(x, x_i^e)^{1+\epsilon_1}} \). there is

\[ \omega([x \in E^c; g_i^k(x) > \lambda]) = \omega([x \in E^c; \frac{\mu(B_i^k)^{1+\epsilon_1}}{V(x, x_i^e)^{1+\epsilon_1}} > \lambda]) \leq \frac{1}{\lambda^{\alpha}}. \]

Application lemma 1,

\[ \omega([x \in E^c; M_0(T_\alpha F_4)(x) > \lambda]) \leq \omega([x \in E^c; \sum_{k=k_0}^{\infty} \sum_{i} C_i^k g_i^k(x) > \lambda]) \]

\[ \leq C\lambda^{-r} \sum_{k=k_0}^{\infty} \sum_{i} (2^k \omega(B_i^k)^{1+\epsilon_1} \cdot \lambda)^r \]

\[ \leq C\lambda^{-r} \sum_{k=k_0}^{\infty} 2^{kr} \left[ \sum_{i} \omega(B_i^k) \right]^{1+\epsilon_1} \lambda^r \]

\[ \leq C(\| f \|_{H_{\omega}^{1+\epsilon_1}(x)/\lambda}^q). \]

Completed the proof of the theorem 4.

Finally, we give the application of the theorem. Nagel and Stein consider a singular integral operator \( \tilde{T} \) which on an unbounded model polynomial region. The operator \( \tilde{T} \) is initially the
mapping form $C_0^\infty(M)$ to $C^\infty(M)$, whose distribution kernel $\tilde{K}(x, y)$ is smooth on the diagonal away from $M \times M$, and the following four conditions are true:

(I-1) If $\varphi, \psi \in C_0^\infty(M)$ have disjoint support, then
\[
\langle \tilde{T}\varphi, \psi \rangle = \int_{M \times M} \tilde{K}(x, y) \varphi(y) \psi(x) \, dx \, dy.
\]

(I-2) If $\varphi$ is a regular bump function associated with a sphere of radius $r$, then
\[
|\partial_x^a \tilde{T}\varphi| \lesssim r^{-a}. \tag{I-2}
\]

More precisely, for each integer $a \geq 0$, there is another integer $b \geq 0$ and constant $M_{a,b}$ such that
\[
\sup_{x \in M} r^a |(\partial_x^a \tilde{T}\varphi)(x)| \leq M_{a,b} \sup_{c \leq b} \sup_{x \in B(x_0, r)} r^c |\partial_x^c \varphi|(x).
\]

(I-3) If $x \neq y$, for any $a \geq 0$,
\[
|\partial^a_{x,y} \tilde{K}(x, y)| \lesssim d(x, y)^{-a} V(x, y)^{-1}. \tag{I-3}
\]

(I-4) while $x$ and $y$ can be exchanged, properties (I-1) to (I-3) also hold. These properties are also true for the conjugate operator defined by $\langle \tilde{T}^c \varphi, \psi \rangle = \langle \tilde{T}\psi, \varphi \rangle$.

**Supplementary Lemma [10-14]**

Lemma 1: Let $g_k$ be a measurable function sequence, let $0 < r < 1$, Suppose $\mu(\{|g_k| > \lambda\}) \leq C/\lambda^r$, where $C$ is independent of $k$ and $\lambda$. Then, for every sequence $\{c_k\}$ in $l^r$, there is
\[
\mu(x: \sum_k c_k g_k > \lambda) \leq \frac{2 - r}{1 - r} \sum_k |c_k|^r.
\]

The following lemma is the Whitney decomposition theorem in the homogeneous space [5].

Lemma 2: Let $\Omega$ be the true open subset of $X$, Let $d(x) = \inf\{d(x, y): y \not\in \Omega\}$. And $r(x) = d(x)/30$. Then there is an $\Omega$ independent of number $L$ and sequence $\{x_k\}_k$, mark $r(x_k)$ as $r_k$, there is

1. $B(x_k, r_k/4)$ are mutually disjointed;
2. $\cup_k B(x_k, r_k) = \Omega$;
3. For each given $k, B(x_k, 15r_k) \subset \Omega$;
4. For each given $k, x \in B(x_k, 15r_k)$, containing $15r_k < d(x) < 45r_k$;
5. For each given $k$, exist $y_k \not\in \Omega$ such that $d(x_k, y_k) < 45r_k$;
6. For each given $k$, the number of non-empty ball $B(x_i, 13r_i)$ does not exceed $L$, these intersecting with $B(x_k, 13r_k)$.

Lemma 3: Let $\Omega$ be an open subset of $X$ with finite measure. Consider sequence of $\{x_k\}_k$ and $\{r_k\}_k$ given by the lemma 2. Then the non-negative function $\{\varphi_k\}_k$ satisfies:

1. For any given $k$, $0 \leq \varphi_k \leq 1$, $\sup_k \varphi_k \subset B(x_k, 2r_k)$ and $\sum_k \varphi_k = 1$;
2. For any given $k$ and $x \in B(x_k, r_k)$, $\varphi_k(x) \geq 1/\Omega$, the constant $\Omega$ independent of $\Omega$;
3. There is a positive number $C$ independent of $\Omega$, which for all $k$ and $\varepsilon \in (0,1]$, $\|\varphi_k\|_{L^1(x_k, \varepsilon)} \leq C \mu(B(x_k, r_k))$.

Lemma 4: Let $\delta > 0, 0 < \alpha \leq 1, 0 < \alpha < \alpha/n$, There is a constant $C$ related to only $C_1$ and $C_3$ that makes:
\[
\int_{d(x, y) \geq \delta} \frac{1}{V'(x, y)^{(1-\alpha)} d(x, y)^{\alpha}} \omega d(y) \leq C \omega(B(x, \delta))^{\alpha} \delta^{-\alpha}
\]

Proof:
\[
\int_{d(x,y)\geq \delta} \frac{1}{V'(x,y)^{(1-\alpha)}d(x,y)^{\alpha}} \omega d(y)
\]
\[
= \sum_{j=0}^{\infty} \int_{2^j\delta \leq d(x,y) \leq 2^{j+1}\delta} \frac{1}{V'(x,y)^{(1-\alpha)}d(x,y)^{\alpha}} \omega d(y)
\]
\[
\leq \sum_{j=0}^{\infty} \left[\omega(B(x,2^j\delta))\right]^{1-\alpha} \frac{(2^j\delta)^{\alpha}}{\omega(B(x,2^{j+1}\delta))}
\]
\[
\leq C_1 C_3 \sum_{j=0}^{\infty} 2^{-j(a-\alpha n)} \omega(B(x,\delta))^{\alpha} \delta^{-\alpha}
\]
\[
\leq C\omega(B(x,\delta))^{\alpha} \delta^{-\alpha}.
\]

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References


