Upper Semicontinuity of Global Attractors for Singularly Perturbed Plate Equations

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Abstract. In the paper, upper semicontinuity of global attractors of singularly perturbed plate equations on an unbounded domain with small positive parameter, is considered. Under suitable assumptions, the equations possess a family of global attractors in natural energy space, and the corresponding singular limit equation, i.e., the parabolic equation, possesses a global attractor, which can be naturally embedded into a compact set of the natural energy space. Using the idea of tail estimates, the author established the upper semicontinuity of the family of global attractors to the compact set in the natural energy space (even more regular space) with respect to the Hausdorff semidistance, as the perturbation parameter tends to zero. The results obtained are new and are not common in the existing literature.

Introduction

Infinite-dimensional dynamical systems generated by evolutionary equations have long been an important subject of research interest of many theoretical or empirical mathematical scientists, and there has been a vast literature to investigate the dynamics of evolutionary models, see e.g., references [1-5] and references therein. Among these research work, the continuity of the global attractors of hyperbolic evolutionary equations under singular perturbation is of a great important topic.

Let us firstly briefly recall relevant research results in the existing literature. In [11] and [12], Hale and Raugel considered the upper semicontinuity and lower semicontinuity of the family of attractors $\mathcal{A}_\varepsilon$ with respect to the topology of $H_0^1(\Omega) \times L^2(\Omega)$ for attractors of singularly perturbed wave equations on bounded domain, respectively. Then Kostin [13] set forth a regular method to deal with the continuity of attractors of abstract hyperbolic equation under singular perturbation which only can be applied to the case of bounded domain. In recent ten years, many authors proceeded to investigates the upper semicontinuity of global attractors $\mathcal{A}_\varepsilon$ of a family of kinds of evolutionary equations, such as perturbed Cahn-Hilliard equations [14-15], parabolic equations [16-17], semilinear wave equations [18-19].

For the case of unbounded domain, It seems that the results about the upper semicontinuity of global attractors are not common in the existing literature. On the basis of [6-7], in [20] the authors studied singularly perturbed wave equations, and concluded that the upper semicontinuity of global attractors is just related to the topology of $H_0^1(\Omega) \times H^{-\alpha}(0 < \alpha \leq 1)$, which is coarser than the one of natural energy space $H_0^1(\Omega) \times L^2(\Omega)$. Also, some authors [21-22] considered the upper semicontinuity of global attractors of evolutionary equations on some spacial unbounded domain.

Problems, Assumptions and Preliminaries

In the paper, we pay attention to initial value problems of plate equations on $\mathbb{R}^n$

$$\varepsilon u_{tt} + \Delta^2 u + u_t + \beta(x)u = f(x, u), \quad x \in \mathbb{R}^n, \quad t \geq 0, \quad (1)$$

$$u(x, 0) = u_0(x), \quad x \in \mathbb{R}^n, \quad (2)$$
\[ u_t(x, 0) = u_1(x), \quad x \in \mathbb{R}^n, \]  
(3)

where \( n \leq 7 \), and its singular limit problems as \( \varepsilon \to 0 \)

\[ u_t + \Delta^2 u + \beta(x)u = f(x, u), \quad x \in \mathbb{R}^n, \quad t \geq 0, \]  
(4)

\[ u(x, 0) = u_0(x), \quad x \in \mathbb{R}^n, \]  
(5)

under following two standing assumptions on the coefficient function \( \beta \) and nonlinear function \( f \)

**Assumption 1.** \( \beta : \mathbb{R}^n \to \mathbb{R} \) is a measurable function with following properties:

(1) For every \( \gamma \), there is a \( C_\gamma \) such that for all \( n \in H^2(\mathbb{R}^n) \),

\[ \int_{\mathbb{R}^n} |\beta(x)||u(x)|^2 dx \leq \gamma \| u \|^2_{H^2(\mathbb{R}^n)} + C_\gamma \| u \|^2_{L^2(\mathbb{R}^n)} ; \]  
(6)

(2) There is \( \lambda_0 > 0 \) such that for all \( n \in H^2(\mathbb{R}^n) \),

\[ \| \Delta u \|^2_{L^2(\mathbb{R}^n)} + \int_{\mathbb{R}^n} \beta(x)|u(x)|^2 dx \geq \lambda_0 \| u \|^2_{L^2(\mathbb{R}^n)} . \]  
(7)

**Assumption 2.** Function \( f : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R} \), \( (x, u) \mapsto f(x, u) \) satisfies \( C^2 \)-Carathéodory condition, i.e., if for every \( u \in \mathbb{R} \) the map \( x \mapsto f(x, u) \) is Lebesgue measurable and for a.e., \( x \in \mathbb{R}^n \) the map \( u \mapsto f(x, u) \) is a function of of \( C^2 \) class. The canonical primitive of \( f \) is defined by

\[ F(x, u) = \int_0^u f(x, s)ds. \]

Moreover \( f \) and \( F \) satisfy the following properties:

(1) \( f(\cdot, 0) \in L^2(\mathbb{R}^n) \), \( \frac{\partial f}{\partial u}(\cdot, 0) \in L^\infty(\mathbb{R}^n) \);

(2) Growth condition, i.e., for a.e., \( x \in \mathbb{R}^n \) and \( u \in \mathbb{R} \),

\[ \left| \frac{\partial^2 f}{\partial u^2}(x, u) \right| \leq C_0(1 + |u|^p), \quad p > 0, \quad p(n-4) < 8-n, \quad n \leq 7 ; \]  
(8)

For positive constants \( C_0 \) and \( p \);

(3) There exists a measure function \( c : \mathbb{R}^n \to \mathbb{R} \), \( c(\cdot) \in L^1(\mathbb{R}^n) \) and a \( \mu > 0 \) such that

\[ f(x, u)u - \mu F(x, u) \leq c(x), \text{ for a.e., } x \in \mathbb{R}^n, \quad u \in \mathbb{R} . \]  
(9)

and

\[ F(x, u) \leq c(x), \text{ for a.e., } x \in \mathbb{R}^n, \quad u \in \mathbb{R} . \]  
(10)

There are particular functions \( \beta \) and \( f \) satisfying all the conditions in two Assumptions above, please refer to [6-8] for more details.

Throughout the paper, except for particular emphasis on the domain, we concisely denote spaces \( L^p(\mathbb{R}^n)(p \geq 1) \), \( W^{s,2}(\mathbb{R}^n)(s \in \mathbb{Z}^+) \) by \( L^p \), \( H^s \) with norms \( \| \cdot \|_{L^p} \), \( \| \cdot \|_{H^s} \) (\( \| \cdot \| \) is particular for \( p = 2 \)) respectively, where \( W^{s,2}(\mathbb{R}^n)(s \in \mathbb{Z}^+) \) are the standard Sobolev spaces and let \( H^{-s} \) be the dual spaces of \( H^s \). Moreover, \( (\cdot, \cdot) \) stands for the inner product in \( L^2(\mathbb{R}^n) \) and \( |\cdot| \) the modular (or absolute value) of \( u \).

We define an operator \( \tilde{A} : D(\tilde{A}) \subset L^2 \to L^2 \) as

\[ \tilde{A}u = \Delta^2 u + \beta u, \]

\[ D(\tilde{A}) = \{ u \in H^2 | \Delta^2 u + \beta u \in L^2 \} = H^4. \]

and have the following assertion.

**Lemma 1.1.** The operator \( \tilde{A} : D(\tilde{A}) \to L^2 \) is self-adjoint and densely defined in \( L^2 \) with \( \text{Re}(\sigma(\tilde{A})) > 0 \), i.e., all the elements of the spectrum of \( \tilde{A} \) have a positive real part, so \( \tilde{A} \) is a sectorial
operator in $L^2$. Moreover, let $X^\alpha = D(\tilde{A}^\alpha)$ be the family of fractional power spaces generated by $\tilde{A}$, then $X^{\frac{1}{2}} = H^2$ and the norm $\|\tilde{A}^{\frac{1}{2}}(\cdot)\|$ defined on the Hilbert space is equivalent to the usual norm.

The proof of the last lemma comes from the Lax-Milgram theorem and an application of Corollary 2.4.10 in [3]. According to Lemma 1.1, we can define the family $X^\alpha = D(\tilde{A}^\alpha)$, $\alpha \in \mathbb{R}$, of fractional power spaces with $X^{-\alpha}$ being the dual of $X^\alpha$ for $\alpha > 0$, be generated by $\tilde{A}$. So we have $X^0 = L^2; X^{\frac{1}{2}} = H^2; X^1 = H^4$. The operator $\tilde{A}$ induces two families of operator, one is $\tilde{A}_\alpha : D(\tilde{A}_\alpha) = X^{\alpha - 1} \rightarrow X^{\alpha - 1}, \tilde{A}_\alpha u = \triangle^2 u + \beta u$, $\alpha \in \mathbb{R}$ with $\tilde{A}_1 = \tilde{A}$, the other is

$$A_\alpha^\alpha : D(A_\alpha^\alpha) = X^{\alpha - 1} \times X^{\frac{3}{2}} \rightarrow X^{\frac{3}{2}} \times X^{\frac{1}{2}}, A_\alpha^\alpha (u, v) = (-v, \frac{1}{\epsilon}(v + \tilde{A}_\alpha u)), \alpha \in \mathbb{R},$$

with $A_1^1 = A$. We know without any difficulty that $\tilde{A}_\alpha$ is a sectorial operator in $X^{\alpha - 1}$ and $-A_\alpha^\alpha$ is an $m$-dissipative operator in $X^{\frac{3}{2}} \times X^{\frac{1}{2}}$ for any $\alpha \in \mathbb{R}$. Also, we have the following basic result.

**Lemma 1.2.** For all $\alpha, \beta \in \mathbb{R}$ with $\beta \geq \alpha$, the map $\varphi_{\beta, \alpha} : X^\beta \rightarrow X^\alpha$ is defined, linear, bounded, and injective. The set $\varphi_{\beta, \alpha}(X^\beta)$ is dense in $X^\alpha$ and $\varphi_{\alpha, \alpha}$ is the identity in $X^\alpha$ for any $\alpha \in \mathbb{R}$. Moreover,

$$\varphi_{\gamma, \alpha} = \varphi_{\beta, \alpha} \circ \varphi_{\gamma, \beta}; \alpha, \beta, \gamma \in \mathbb{R}, \gamma \geq \beta \geq \alpha,$$

for all $\alpha, \gamma \in \mathbb{R}, \theta \in [0, 1]$ and $x \in X^\gamma$ with $\alpha \leq \gamma$ and $\beta = (1 - \theta)\alpha + \theta\gamma$, the interpolation inequality

$$\|\varphi_{\gamma, \alpha}x\|_{X^\alpha} \leq \|\varphi_{\gamma, \alpha}\|_{X^\alpha}^{1 - \theta} \cdot \|x\|_{X^\gamma}^\theta$$

holds.

For a complete understanding of fractional power space induced by a sectorial operator $A$ with $\text{Re}(\sigma(A)) > 0$, we refer readers to [2] for more details.

So we have the corresponding Cauchy problem associated with the problem (1)-(3)

$$\frac{dw_\epsilon}{dt} + A_\epsilon w_\epsilon = \tilde{f}(w_\epsilon(t)), t \geq 0, \quad w_\epsilon(0) = w_0.$$  \hspace{1cm} (11)

Where

$$w_\epsilon(t) = (w_\epsilon(t), \frac{dw_\epsilon}{dt}(t)), \tilde{f}(w_\epsilon(t)) = (0, \frac{1}{\epsilon}f(\cdot, w_\epsilon(t))), w_0 = (w_0, u_0) \in H^2 \times L^2.$$

Under Assumptions 1 and Assumptions 2, it was proved in [9] that the strongly continuous semigroup $\{S_\epsilon^\tau\}_{\epsilon \geq 0}$ associated with the problem (1)-(3) has a global attractor $\mathcal{A}_\epsilon$ in phase space $H^2 \times L^2$. Furthermore, from [10], if the nonlinear function $f$ has subcritical exponent, then, for any $\epsilon > 0$, the attractor $\mathcal{A}_\epsilon$ established in [9] has the following regularity: $\mathcal{A}_\epsilon \subset H^4 \times H^2$ and is bounded subset of $H^4 \times H^2$. We summarize the result in the following theorem.

**Theorem 1.1.** The strongly continuous semigroup $\{S_\epsilon^\tau\}_{\epsilon \geq 0}$ associated with the problem (1)-(3) has a global attractor $\mathcal{A}_\epsilon$. Moreover, for any $\epsilon > 0$, $\mathcal{A}_\epsilon \subset H^4 \times H^2$ and is bounded in $H^4 \times H^2$.

Also, as for the corresponding Cauchy problem associated with the singular limit problem (4)-(5)

$$\frac{du}{dt} + \tilde{A}u = \tilde{f}(u), \quad t \geq 0, \quad u(0) = u_0.$$  \hspace{1cm} (13)

in [8], we also have showed that the semigroup $\{\tilde{S}_\epsilon^\tau\}_{\epsilon \geq 0}$ admits a global attractor $\tilde{\mathcal{A}}$ and it is compact subset of $D(\tilde{A})$. We restate the result as follows.

**Theorem 1.2.** The strongly continuous semigroup $\{\tilde{S}(t)\}_{\epsilon \geq 0}$ associated with the problem (4)-(5) has a global attractor $\tilde{\mathcal{A}}$ in $H^2$. Moreover, $\tilde{\mathcal{A}}$ is compact invariant in $H^4$. 

24
The Main Results and Remarks

In view of the last theorem, we can define the set

$$
\mathcal{A}_0 := \{(u, v) | u \in \tilde{A}, \ v = -\Delta^2 u - \beta u + f(x, u)\}.
$$

The set $\mathcal{A}_0$ is a compact set in $H^2 \times L^2$, and is a natural embedding into $H^2 \times L^2$ of the attractor $\tilde{\mathcal{A}} \subset H^2$.

For any two bounded sets $A$ and $B$ in Banach space $X$, we define the Hausdorff semidistance by

$$
\delta_X(A, B) = \sup_{x \in A} \left( \inf_{y \in B} \|x - y\|_X \right).
$$

Now, we can state the main result of the paper as following.

**Theorem 1.3.** The family of attractors $\mathcal{A}_\varepsilon$ is upper-semicontinuous at $\varepsilon = 0$ with respect to the topology $X^{1-\alpha_1} \times X^{\alpha_2}$ for any $0 < \alpha_1 \leq \frac{1}{2}$, $-\frac{1}{2} \leq \alpha_2 < \frac{1}{2}$, i.e.,

$$
\lim_{\varepsilon \to 0^+} \delta_{X^{1-\alpha_1} \times X^{\alpha_2}}(\mathcal{A}_\varepsilon, \mathcal{A}_0) = 0. \tag{15}
$$

In particular, we have

$$
\lim_{\varepsilon \to 0^+} \delta_{H^2 \times L^2}(\mathcal{A}_\varepsilon, \mathcal{A}_0) = 0. \tag{16}
$$

**Remark 2.1.** The equations (15) and (16) imply that we established the upper semicontinuity of the global attractors of the problem (1)-(3) with respect to the topology which is finer than the one of the natural energy space $H^2 \times L^2$.

**Remark 2.2.** In the proof of the Theorem 1.3, we shall confront two main difficulties. One difficulty is that in the case of unbounded domain, the Sobolev embedding $H^2 \hookrightarrow L^2$ is no longer compact, and this difficulty not only gives us trouble in proving the existence of attractors $\mathcal{A}_\varepsilon$ and $\tilde{\mathcal{A}}$, but also prevents us from using Arzelá-Ascoli Theorem to obtain uniform convergence of a sequence of solutions; the other difficulty is to establish the desired uniform $H^4 \times H^2$-estimates in $r$ of the attractors $\mathcal{A}_\varepsilon$. As we know from [11] that there must be some estimates of high regularity enough (such as $H^4 \times H^2$-estimates) of the solutions for $S_r$ to guarantee upper semicontinuity of $\mathcal{A}$ associated with the topology of natural energy space $H^2 \times L^2$. Using the method, in [9] we established the tail-estimates of the solutions to overcome the first difficulty above. Also, motivated by the idea in [11], we can establish uniform $H^4 \times H^2$-estimates in $\varepsilon$ of the attractors $\mathcal{A}_\varepsilon$ by introducing a very useful functional to overcome the second difficulty.

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References


