System of Split General Variational Inequality Problems in Semi-inner Product Spaces

Ya-li ZHAO* and Xin LIU

College of Mathematics and Physics, Bohai University, Jinzhou, Liaoning 121013, China

*Corresponding author

Keywords: System of split general variational inequality problems, Sunny retraction mapping, Semi-inner product spaces, Generalized adjoint operator.

Abstract. In this paper, we introduce a new system of split general variational inequality problems which is a natural extension of system of split variational inequality problems and split variational inequality problems in semi-inner product spaces. We employ the retraction technique to propose an iterative algorithm for computing the approximate solution of the system of split general variational inequality problems. Moreover, the convergence analysis of the iterative algorithm is also discussed. Several special case which can be obtained from the main results are also presented.

Preliminaries

We recall the following concepts and results, which are needed to define the problem and to prove the main result.

Definition 1 Let \( V \) be a vector space over the field \( R \) of real numbers. A functional \( [\cdot, \cdot] : V \times V \to R \) is called a semi-inner product if it satisfies the following conditions:

1. \( [u + v, w] = [u, w] + [v, w] \), \( \forall u, v, w \in V \);
2. \( [\lambda u, v] = \lambda [u, v] \);
3. \( [u, u] > 0 \), for \( u \neq 0 \);
4. \( [u, u] \leq [u, u][v, v] \), \( \forall u, v \in V \).

The pair \((V, [\cdot, \cdot])\) is called a semi-inner product space. As it is observed in [1] that \( \|u\| = \sqrt{[u, u]} \), \( \forall u \in V \), is a norm on \( V \). Hence every semi-inner product space is a normed linear space. On the other hand, in a normed linear space, one can generate semi-inner product in infinitely many different ways. Further, it is noted that a Hilbert space \( H \) can be made into a semi-inner product space, which a semi-inner product is an inner product if and only if the norm it induced verifies the parallelogram law. Let \( Y \) be a semi-inner product space and let \( T : V \to Y \) be an arbitrary operator.

Definition 2 The generalized adjoint operator \( T^+ \) of an operator \( T \) is defined as follows: The domain \( D(T^+) \) of \( T^+ \) consists of those \( y \in Y \) for which there exists \( z \in V \) such that \( [Tx, y]_V = [x, z]_V \) for each \( x \in V \) and \( z = T^+ y \).

Remark 3 As it is observed in [3] that if \( V \) and \( Y \) are Hilbert spaces then the generalized adjoint operator is the usual adjoint operator. In general, \( T^+ \) is not linear even for \( T \) is a bounded linear operator.

In 1967, Giles [5] proved that if the underlying semi-inner produce space \( V \) is a uniformly Convex smooth Banach space then it is possible to define a semi-inner product uniquely which has the following properties:

(i) \( [u, v] = 0 \) for some \( u, v \in V \) if and only if \( v \) is orthogonal to \( u \).
(ii) The semi-inner product is continuous, i.e., for each \( u, v \in V \), we have \( [v, u + \lambda v] \to [u, v] \) as \( \lambda \to 0 \).
(iii) \( [u, \lambda v] = \lambda [u, v] \), \( \forall \lambda \in R \), \( \forall u, v \in V \), i.e., the semi-inner product is with the homogeneity property.
(iv) Generalized Riesz representation theorem: If \( f \) is continuous linear functional on \( V \) then
there is a unique vector $v \in V$ such that $f(u) = [u, v], \forall u \in V$.

Now, we give the properties of the generalized adjoint operator.

**Proposition 4** [2] Let $X$ and $Y$ be 2-uniformly convex smooth Banach space and let $T : X \to Y$ be a bounded linear operator. Then

(i) $D(T^*) = Y$; (ii) $T^*$ is bounded, and to holds that $\|T^*y\| \leq \|T\|\|y\|, \forall y \in Y$.

**Definition 5** [7] Let $D$ be a subset of $C$ and $Q_c$ be a mapping of $C$ into $D$. Then $Q_c$ is said to be sunny if $Q_c(Q_u + t(u - Q_cu)) = Q_cu$, whenever $Q_cu + t(u - Q_cu) \in C$ for $u \in C$ and $t \geq 0$.

**Definition 6** [6] A subset $D$ of $C$ is called a sunny nonexpansive retract of $C$ if there exists a sunny nonexpansive retraction from $C$ into $D$.

**Proposition 7** [6] Let $E$ be a smooth Banach space and let $C$ be a nonempty subset of $E$. Let $Q_c : E \to C$ be a retraction. Then the following are equivalent:

(i) $Q_c$ is sunny and nonexpansive; 
(ii) $\|Q_cu - Q_cv\| \leq \langle u - v, J(Q_cu - Q_cv) \rangle, \forall u, v \in E$; 
(iii) $\langle u - Q_cu, J(v - Q_cu) \rangle \leq 0, \forall u \in E, v \in C$.

**Lemma 8** [4] Let $\rho > 1$ be a real number and $E$ be a smooth Banach space. Then the following statements are equivalent:

(i) $E$ is 2-uniformly smooth; (ii) There is a constant $c > 0$ such that for every $u, v \in E$, the following inequality holds: $\|u + v\|^2 \leq \|u\|^2 + 2\langle u, J(v) \rangle + c\|v\|^2$.

**Remark 9** (a) It follows from [1,5,7] that an normed linear space is a semi-inner product space. In fact, by Hahn Banach theorem, for each $v \in E$ there exists a functional $f_u \in E^*$ such that $\langle u, f_u \rangle = \|u\|^2$. Given any such mapping $f$ from $E$ into $E^*$, we can verify that $\langle v, f_u \rangle$ defines a semi-inner product. Hence, we write the inequality given in Lemma 8 as $\|u + v\|^2 \leq \|u\|^2 + 2\langle v, u \rangle + c\|v\|^2, \forall u, v \in E$. The constant $C$ is chosen with best possible minimum value. We call $C$ as the constant of smoothness of $E$.

(b) The inequalities given in proposition 7(ii) and (iii) can be written as

(ii) $\|Q_cu - Q_cv\| \leq \|u - v, Q_cu - Q_cv\|, \forall u, v \in E$; (iii) $\langle u - Q_cu, v - Q_cu \rangle \leq 0, \forall u \in E, v \in C$.

In the following, otherwise special statement for each $i \in \{1,2\}$, suppose that $Q_{C_i} : E_i \to C_i$ is a sunny and nonexpansive retraction, where $E$ is a smooth Banach space and $C_i$ be a nonempty subset of $E_i$. Let $E_i$ and $E_2$ be 2-uniformly convex, smooth Banach spaces and for each $i \in \{1,2\}$, let $C_i \subset E_i$ be a nonempty closed and convex set and let $J_i : E_i \to 2^{E_i}$ and $J_j : E_j \to 2^{E_j}$ be the normalized duality mappings. Let $F_i, f_i, G_i, g_i : C_i \to E_i$ be nonlinear mappings, and let $A : E_i \to E_i$ be a bounded linear operator. We introduced the following system of split general variational inequality problems (in short, $SS_{\rho}GVIP$): Find $(u, v) \in C_1 \times C_2$ such that

\[ \langle \lambda F_i v_1 + u_i - f_i(v_i), J_i(w_i - u_i) \rangle \geq 0, \forall w_i \in C_1; \langle \lambda G_i u_i + v_i - g_i(u_i), J_i(w_i - v_i) \rangle \geq 0, \forall w_i \in C_1, \]

and such that $(u_1, v_1)$ with $u_2 = Au_1, v_2 = Av_1 \in C_2$ solves

\[ \langle \lambda F_2 v_2 + u_2 - f_2(v_2), J_2(w_2 - u_2) \rangle \geq 0, \forall w_2 \in C_2; \langle \lambda G_2 u_2 + v_2 - g_2(u_2), J_2(w_2 - v_2) \rangle \geq 0, \forall w_2 \in C_2, \]

for any $\lambda, \gamma > 0$. Above $SS_{\rho}GVIP$ is equivalent to find $(u_1, v_1) \in C_1 \times C_2$ such that

\[ [\lambda F_i v_1 + u_i - f_i(v_i), w_i - u_i] \geq 0, \forall w_i \in C_1; \]

\[ [\lambda G_i u_i + v_i - g_i(u_i), w_i - v_i] \geq 0, \forall w_i \in C_i; \]
and such that \((u_2, v_2)\) with \(u_2 = Au_1 \in C_2, v_2 = Av_1 \in C_2\) solves

\[
[\gamma F_2 v_2 + u_2 - f_2(v_2), w_2 - u_2] \geq 0, \forall w_2 \in C_2;
\]

\[
[\gamma G_2 u_2 + v_2 - g_2(u_2), w_2 - v_2] \geq 0, \forall w_2 \in C_2;
\]

for any \(\lambda, \gamma > 0\). If for each \(i \in \{1, 2\}, f_i, g_i = I_i\), where \(I_i\) is identity mapping on \(C_i\), then \(SS\_p\_GVIP\) (1)-(4) reduced to the following system of \(SS\_p\_GVIP\) introduced and studied by Kazmi and Furkan [9]: Find \((u_i, v_i) \in C_i \times C_i\) such that

\[
[\lambda F_1 v_i + u_i - v_i, w_i - u_i] \geq 0, \forall w_i \in C_1;
\]

\[
[\lambda G_1 u_i + v_i - u_i, w_i - v_i] \geq 0, \forall w_i \in C_1;
\]

and such that \((u_2, v_2)\) with \(u_2 = Au_1 \in C_2, v_2 = Av_1 \in C_2\) solves

\[
[\gamma F_2 v_2 + u_2 - v_2, w_2 - u_2] \geq 0, \forall w_2 \in C_2;
\]

\[
[\gamma G_2 u_2 + v_2 - v_2, w_2 - u_2] \geq 0, \forall w_2 \in C_2;
\]

for any \(\lambda, \gamma > 0\). It is worth mentioning that Kazmi and Furkan in [8] firstly introduced and studied \(SS\_p\_VIP\) in the setting Banach spaces. They used the retraction technique to propose and analyze an iterative algorithm for computing the approximate solution of \(SS\_p\_VIP\) (5-8) in 2-uniformly convex smooth Banach spaces. Inspired and motivated by the recent research work [8,10-12] in this paper, we introduced and study a new system of split general variational inequality problems(\(SS\_p\_VIP\) (1)-(4)) employing the retraction technique, we propose an iterative algorithm with errors for computing the approximate solution of \(SS\_p\_VIP\) (1)-(4) in 2-uniformly convex smooth Banach spaces. Further, convergence analysis of the iterative algorithm is discussed. Several special cases which can be obtained from the main result are also presented. The problems and the results obtained here are new and different from the existing problems and results in the literature.

**Iterative Algorithms**

By making use of proposition 7, we noted that \(SS\_p\_VIP\) (1)-(4) can be formulated as follows: Find \((u_1, v_1) \in C_1 \times C_1\) with \((u_2, v_2) = (Au_1, Av_1) \in C_2 \times C_2\) such that

\[
u_1 = Q_{C_1}(v_1 - \lambda F_1 v_1),
\]

\[
u_2 = Q_{C_2}(v_2 - \gamma F_2 v_2),
\]

\[
u_2 = Q_{C_2}(u_2 - \gamma G_2 u_2),
\]

for any \(\lambda, \gamma > 0\). Based on above arguments, we propose the following iterative algorithm for approximating a solution to \(SS\_p\_VIP\) (1) - (4). Let \(\{\alpha^n\} \subseteq (0,1)\) be a sequence such that \(\sum_{n=1}^{\infty} \alpha^n = \infty\).

**Algorithm 10** Given \((u_1^0, v_1^0) \in C_1 \times C_1\), compute the iterative sequence \(\{(u_1^n, v_1^n)\}\) defined by the iterative schemes:
\[ p_i^n = Q_{C_i}(f_i(v^n_i) - \lambda F_i v^n_i) + \epsilon_i^n, \quad (13) \]

\[ q_i^n = Q_{C_i}(g_i(u^n_i) - \lambda G_i u^n_i) + \epsilon'_i^n, \quad (14) \]

\[ p_2^n = Q_{C_2}(f_2(v^n_2) - \gamma F_2 v^n_2) + \epsilon_2^n, \quad (15) \]

\[ q_2^n = Q_{C_2}(g_2(u^n_2) - \gamma G_2 u^n_2) + \epsilon'_2^n, \quad (16) \]

\[ u_n^{n+1} = (1 - \alpha^n)u^n_i + \alpha^n(p^n_i + \rho A^*(p^n_2 - Ap^n_1)), \quad (17) \]

\[ v_i^{n+1} = (1 - \alpha^n)v_i^n + \alpha^n(q_i^n + \rho A^*(q_2^n - Aq_1^n)), \quad (18) \]

For all \( n = 0,1,2, \ldots \) and \( \lambda, \gamma, \rho > 0 \), where \( A^* \) is the generalized adjoint operator of \( A \), and \( u_2^n = A u_1^n \), \( v_2^n = A v_1^n \) for all \( n \), and for each \( i \in \{1,2\} \), \( \{e_i^n\}_{n=0}^\infty \subset E_i \) and \( \{\epsilon_i^n\}_{n=0}^\infty \subset E_2 \) are four sequence to take into account of a possible inexact computation satisfying the following conditions:

\[ \lim_{n \to \infty} \epsilon_i^n = \lim_{n \to \infty} \epsilon_i^n = 0. \]

**Algorithm 11** Given \((u_1^0,v_1^0) \in C_1 \times C_2\), compute the iterative sequence \(\{(u_i^n,v_i^n)\}\) defined by the iterative schemes:

\[ p_i^n = Q_{C_i}(v_i^n - \lambda F_i v_i^n), \quad q_i^n = Q_{C_i}(u_i^n - \lambda G_i u_i^n), \quad p_2^n = Q_{C_2}(v_2^n - \gamma F_2 v_2^n), \quad q_2^n = Q_{C_2}(u_2^n - \gamma G_2 u_2^n), \]

\[ u_n^{n+1} = (1 - \alpha^n)u_i^n + \alpha^n(p^n_i + \rho A^*(p^n_2 - Ap^n_1)), \quad v_i^{n+1} = (1 - \alpha^n)v_i^n + \alpha^n(q_i^n + \rho A^*(q_2^n - Aq_1^n)), \]

for all \( n = 0,1,2, \ldots \) and \( \lambda, \gamma, \rho > 0 \), where \( A^* \) is the generalized adjoint operator of \( A \), and \( u_2^n = A u_1^n \), \( v_2^n = A v_1^n \) for all \( n \).

**Remark 12** Algorithm 11 was proposed by Kamiz and Furkan in [9] to compute the approximate solution of \( SS_pVIP(9)-(12) \), which is a special case of Algorithm 10.

**Main Results**
In order to obtain our main results, we used the following concepts and lemma.

**Definition 13** Let \( F, g : E_1 \to E_i \) be two mappings. \( F \) is said to be

1. \( \alpha \) – strongly monotone with respect to \( g \), if there exists a constant \( \alpha > 0 \) such that

\[ [Fu_i - Fu_2, gu_i - gu_2] \geq \alpha \|u_i - u_2\|^2, \quad \forall u_i, u_2 \in E_i. \]

2. \( \beta \) – Lipschitz continuous, if there exists a constant \( \beta > 0 \) such that

\[ \|Fu_i - Fu_2\| \leq \beta \|u_i - u_2\|, \quad \forall u_i, u_2 \in E_i. \]

**Lemma 14[9]** Let \( \{a_n\}_{n=0}^\infty \), \( \{b_n\}_{n=0}^\infty \) and \( \{c_n\}_{n=0}^\infty \) be nonnegative sequences satisfying

\[ a_{n+1} \leq (1 - \lambda_n)a_n + \lambda_n b_n + c_n, \quad \forall n \geq 0, \]

where \( \{\lambda_n\}_{n=0}^\infty \subset [0,1], \) \( \sum_{n=0}^\infty \lambda_n = \infty \), \( \sum_{n=0}^\infty c_n < \infty \), \( \lim_{n \to \infty} b_n = 0 \). Then \( \lim_{n \to \infty} a_n = 0. \)

Now, we prove that the sequence of approximate solution of \( SS_pGVIP(1)-(4) \) generated Algorithm 10 converges strongly to the solution of \( SS_pGVIP(1)-(4) \).
Theorem 15 For each $i \in \{1,2\}$, let $C_i$ be a nonempty closed and convex subset of 2-uniformly convex smooth Banach space $E_i$ with constant of smoothness $C_i$. Let $F_i : C_i \to E_i$ be $\alpha_i$ - strongly monotone with respect to $f_i : C_i \to E_i$ and $\beta_i$ - Lipschitz continuous, let $G_i : C_i \to E_i$ be $\alpha_j$ - strongly monotone with respect to $g_i : C_i \to E_i$ and $\beta_j$ - Lipschitz continuous, let $F_2 : C_2 \to E_2$ be $\delta_i$ - strongly monotone with respect to $g_2$ and $\eta_i$ - Lipschitz continuous, and let $f_i, g_i : C_i \to E_i$ be $\xi_i, \zeta_i$ - Lipschitz continuous, respectively, and let $f_2, g_2 : C_2 \to E_2$ be $\zeta_1, \zeta_2$ - Lipschitz continuous, respectively. Let $A : E_1 \to E_2$ be bounded linear operator. Suppose that $(u_i, v_i) \in C_i \times C_i$ be a solution to $SS_p, GVIP$ (1)-(4), then the sequence $\{(u^n_i, v^n_i)\}$ generated by Algorithm 10 converges strongly to $(u_i, v_i)$ provided that the constant $\lambda > 0$ satisfies the following conditions:

\[
\max_{i=1,2}\left\{\alpha_i - \frac{\sqrt{\alpha_i^2 - C_i\beta_i^2(\xi_i^2 - \alpha_i^2)}}{C_i\beta_i}\right\} < \min_{i=1,2}\left\{\alpha_i + \frac{\sqrt{\alpha_i^2 - C_i\beta_i^2(\xi_i^2 - \alpha_i^2)}}{C_i\beta_i}\right\};
\]
\[
\alpha_i > \beta_i \sqrt{C_{i}(\xi_i^2 - \alpha_i^2)}, \xi_i > \alpha_i, \alpha_i = \frac{1-m\theta_{2}}{1+m}, m = \rho \|A^*\|\|A\|
\]
\[
\theta_{2} = \sqrt{\xi_i^2 - 2\gamma \sigma_i} + C_i \gamma^2 \eta_i^2, \gamma > 0, \rho > 0.
\]

**Proof.** Given that $(u_i, v_i)$ is a solution of $SS_p, GVIP$ (1)-(4), that is, $u_i, v_i$ satisfy the relations (9)-(13). Since $F_i : C_i \to E_i$ is $\alpha_i$ - strongly monotone with respect to $f_i$ and $\beta_i$ - Lipschitz continuous and $f_i : C_i \to E_i$ is $\xi_i$ - Lipschitz continuous, from Algorithm 10 (13) and (9), we have

\[
\|P^n_i - u_i\| = \|Q_{C_i}(f_i(v_i^n) - \lambda F_i v_i^n) + e_i - Q_{C_i}(f_i(v_i) - \lambda F_i v_i)\|
\leq \|f_i(v_i^n) - f_i(v_i)\|^2 + 2\lambda[F_i v_i^n - F_i v_i, f_i(v_i^n) - f_i(v_i)] + c_i \lambda^2 \|F_i v_i^n - F_i v_i\|^2 + \|e_i^n\|
\leq \theta_i \|v_i^n - v_i\| + \|e_i^n\|
\]

where $\theta_i = \sqrt{\xi_i^2 - 2\lambda \alpha_i} + C_i \lambda^2 \beta_i^2$. Next, since $G_i : C_i \to E_i$ is $\alpha_j$ - strongly monotone with respect to $g_i$ and $\beta_j$ - Lipschitz continuous and $g_i : C_i \to E_i$ is $\xi_j$ - Lipschitz continuous, from Algorithm 10 (18) and (14), we get

\[
\|g^n_i - v_i\| \leq \theta_2 \|v_i^n - v_i\| + \|e_i^n\|
\]

where $\theta_2 = \sqrt{\xi_i^2 - 2\alpha_j \lambda^2} + C_i \lambda^2 \beta_i^2$. Again, since $F_2 : C_2 \to E_2$ is $\sigma_i$ - strongly monotone with respect to $f_2 : C_2 \to E_2$ and $\eta_i$ - Lipschitz continuous and $f_2 : C_2 \to E_2$ is Lipschitz, from Algorithm 10 (15) and (11), we have

\[
\|P^n_2 - u_2\| \leq \theta_3 \|v_2^n - v_2\| + \|e_2^n\|
\]

where $\theta_3 = \sqrt{\xi_2^2 - 2\lambda \sigma_i} + C_i \lambda^2 \beta_i^2$. Since $G_2 : C_2 \to E_2$ is $\sigma_j$ - strongly monotone with respect to $g_2 : C_2 \to E_2$ and $\eta_i$ - Lipschitz continuous and $g_2 : C_2 \to E_2$ is $\xi_2$ - Lipschitz continuous, from Algorithm 10 (16) and (12), we have

\[
\|g^n_2 - v_2\| \leq \theta_4 \|u_2^n - u_2\| + \|v_2^n\|
\]

where $\theta_4 = \sqrt{\xi_2^2 - 2\sigma_j \lambda^2} + C_i \lambda^2 \beta_i^2$. Now, using the fact $A^*$ is bounded, we have

\[
\|u_1^n - u_n\| \leq (1 - \alpha_n) \|u^n - u_i\| + \alpha^n \|p^n_1 - u_i + \rho A^*(p^n_1 - Ap^n_i)\|
\leq (1 - \alpha_n) \|u^n - u_i\| + \alpha^n (\theta_i + \rho \|A^*\|\|A\|\|\theta_i\|) \|v_i^n - v_i\| + \alpha^n (1 + \rho \|A^*\|\|A\|\|\xi_i^n\|) + \rho \|A^*\|\|\xi_i^n\|.
\]
Similarly, we obtain

\[
\begin{aligned}
\left\| v^{n+1}_i - v_i \right\| & \leq (1 - \alpha_i) \left\| v^n_i - v_i \right\| + \alpha_i \left( \theta_i + \rho \left\| A \right\| \left\| (\theta_i + \theta_i) \right\| \right) \| u^n_i - u_i \|
+ \alpha_i \left( (1 + \rho \left\| A \right\| \right) \| \epsilon^{n+1}_i \| + \rho \left\| A\right\| \| \epsilon^n_i \|
\end{aligned}
\] (25)

Now, we define the norm on \( E_1 \times E_2 \) by \( \|(u,v)\|_n = \|u\| + \|v\| \), \( \forall (u,v) \in E_1 \times E_2 \). We can easily know that \( (E_1 \times E_2, \| \cdot \|_n) \) is a Banach space. Combining (24) and (25), we have

\[
\left\| (u_i^{n+1}, v_i^{n+1}) - (u_i, v_i) \right\| 
\leq (1 - \alpha^n) \left\| (u_i^n - u_i) + \alpha^n \max \{ k_1, k_2 \} \right\| u_i^n - u_i \| + \| v_i^n - v_i \|
\]

where \( \theta = \max \{ k_1, k_2 \}, \theta_i = \theta_i + m(\theta_i + \theta_i), k_2 = \theta_i + m(\theta_i + \theta_i), \Delta = \max \{ k_1, k_2 \}, k_3 = 1 + m, k_4 = \frac{m}{\| A \|}, m = \rho \left\| A \right\| \| A \|
\]

By condition (19) on \( \lambda \), we know that \( \theta \in (0,1) \), letting \( a_n = \left\| (u_i^n, v_i^n) - (u_i, v_i) \right\|, b_n = \Delta \left\| \epsilon_i^n + \epsilon_i^n \right\| + \| \epsilon_i^n \| \). Since \( \sum_{n=1}^{\infty} a_n = \infty \) and \( \theta \in (0,1) \), then \( \alpha^n (1 - \theta) \in (0,1) \) and \( \sum_{n=1}^{\infty} a_n (1 - \theta) = \infty \), \( \lim_{n \to \infty} b_n = \lim_{n \to \infty} \Delta \left( \| \epsilon_i^n \| + \| \epsilon_i^n \| + \| \epsilon_i^n \| \right) = 0 \).

\( \sum_{n=1}^{\infty} c_n = \infty \), it follows from Lemma 14 and the above inequality that, the whole conditions of Lemma 14 hold. So \( \left\| (u_i^{n+1}, v_i^{n+1}) - (u_i, v_i) \right\| \to 0 \) \( (n \to \infty) \), that is, \( \{ (u_i^{n+1}, v_i^{n+1}) \} \) converges strongly to \( (u_i, v_i) \) as \( n \to \infty \), implying that \( u_i^n \to u_i \) and \( v_i^n \to v_i \) as \( n \to \infty \). Further, it follows from (20) and (21), respectively, that \( p_i^n \to u_i \) and \( q_i^n \to v_i \) as \( n \to \infty \). Hence, it follows from (22) and (23), respectively, that \( p_i^n \to u_i = Au_i \) and \( q_i^n \to v_i = Av_i \) as \( n \to \infty \). This completes the proof.

**Corollary 16** For each \( i \in \{1,2\} \), let \( C_i, E_i, F_i, G_i, A \) be same as in Theorem 15. Suppose that \( (u_i, v_i) \in C_i \times C_2 \) be a solution to \( SS_{\rho}GVIP \) (5-8), then the sequence \( \{ (u_i^n, v_i^n) \} \) generated by Algorithm 11 converges strongly to \( (u_i, v_i) \) provided that the constant \( \lambda > 0 \) satisfying the following conditions:

\[
\max \{ \frac{\alpha_i - \sqrt{\alpha_i^2 - c_i \beta_i (i - a_i^2)}}{c_i \beta_i^2}, \frac{\alpha_i + \sqrt{\alpha_i^2 - c_i \beta_i (1 - a_i^2)}}{c_i \beta_i^2} \} < \lambda < \min \{ \frac{\alpha_i - \sqrt{\alpha_i^2 - c_i \beta_i (i - a_i^2)}}{c_i \beta_i^2}, \frac{\alpha_i + \sqrt{\alpha_i^2 - c_i \beta_i (1 - a_i^2)}}{c_i \beta_i^2} \}; \]

\[
\alpha_i > \beta \sqrt{c_i (1 - a_i^2)}, a_i = \frac{1 + m \theta_i + i}{1 + m}, m = \rho \left\| A \right\| \| A \| \theta_{2;n} = \sqrt{1 - 2 \gamma \sigma_i + c_i \gamma^2 \eta_i^2}, \gamma > 0, \rho > 0. \]

**Acknowledgement**

This research was financially supported by the National Science Foundation (11371070).

**References**


