Frontier Visualization for Nonconvex Models with Increasing and Decreasing Returns to Scale with the Use of Enumeration Methods

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Abstract. A lot of scientific literature in the world is devoted to the returns-to-scale (RTS) evaluation of production units. Kerstens and Vanden Eeckaut were the first who considered the notion and evaluation of RTS for nonconvex Free Disposal Hull (FDH) models. Their methods are based on the comparison of the radial efficiency scores of a production unit in the FDH model and the corresponding non-increasing and non-decreasing RTS reference models. At the same time Cesaroni, Kerstens and Van de Woestyne noted the absence of papers in the world literature devoted to methods of frontier visualization in nonconvex models. In this paper, we developed methods for visualization of production function of nonconvex models and of the corresponding non-increasing and non-decreasing models. Our approach of frontier visualization can be used for decision making based on nonconvex models, it allows us to evaluate RTS for FDH models. Our results are confirmed by computational experiments in applications with real-life data sets.

Introduction

There is a lot of scientific literature in the world devoted to the returns-to-scale (RTS) evaluation of production units. Data Envelopment Analysis (DEA) models and Free Disposal Hull (FDH) models appeared almost simultaneously in papers [1, 3, 4], respectively, for performance management in complex systems. In DEA models, the production possibility set is a convex set. For this reason, solution methods and software have been widely used for modeling and computations of various indicators of unit’s behavior, including such an important indicator as RTS. On the contrary, the production possibility set of FDH models is a non-convex set, for this reason, the development of these models was refrained. Apparently, Kerstens and Vanden Eeckaut [7] were the first to propose a subtle approach for estimating RTS of production units in FDH models. Their approach is based on the comparison of the input radial efficiency of a unit in the FDH model and the corresponding constant, non-increasing and non-decreasing RTS reference technologies. Such an approach requires solving nonlinear mixed integer optimization problems. In [17], it was proved that these problems can be rewritten as mixed integer linear problems, at the same time the size of such problems increases significantly, what complicates the solution process by these
methods. Soleimani-damaneh et al. [19] proposed enumeration algorithm for estimating RTS in FDH models. Leleu [13, 14] suggested a linear programming approach to the same problem. At the same time, the author noted that the proposed linear programming problems do not have any computational advantages over the nonlinear problems, although the former might be easier to program in real-life applications. There have been three papers [19, 20, 21] that simplify the computations needed to implement the method of Kerstens and Vanden Eeckaut [7] to estimate RTS. Podinovski [16] is the first to indicate that the method of Kerstens and Vanden Eeckaut for estimating global returns to scale for nonconvex technologies is incomplete. Moreover, it was shown in [18] that one must distinguish the fourth type of global sub-constant RTS in addition to the three traditional cases (constant, decreasing and increasing returns to scale). Some works are devoted to visualization of Pareto sets in general optimization problems, see, for example, [15].

In this paper, we develop an approach for visualization of FDH frontier with increasing and decreasing returns to scale. Our approach allows us to evaluate returns to scale for FDH models, which requires fewer computations than existing methods.

This paper is structured as follows. Section 2 provides some basic definitions of the convex and less widely applied nonconvex technologies. Section 3 presents two algorithms of frontier visualization for non-convex models with variable, increasing and decreasing returns to scale with the use of enumeration methods. Then follows Section 4 with some computational results. Section 5 concludes.

Preliminaries

Consider a set of $n$ observation of actual production units $(X_j, Y_j)$, $j = 1, \ldots, n$, where the vector of outputs $Y_j = (y_{1j}, \ldots, y_{rj}) > 0$, $j = 1, \ldots, n$ is produced from the vector of inputs $X_j = (x_{1j}, \ldots, x_{mj}) > 0$, $j = 1, \ldots, n$. The traditional FDH technology described in [4] is written as follows:

$$T_{FDH} = \left\{ (X, Y) \mid n \sum_{j=1}^{n} X_j \lambda_j \leq X, \sum_{j=1}^{n} Y_j \lambda_j \geq Y, \sum_{j=1}^{n} \lambda_j = 1, \lambda_j \in \{0, 1\}, j = 1, \ldots, n \right\}, \quad (1)$$

where $\lambda_j, j = 1, \ldots, n$ are integer variables taking on values 0 or 1.

In addition, we consider the constant, non-increasing, non-decreasing returns-to-scale reference technologies associated with the technology $T_{FDH}$. They are written as follows.

$$T_{CRS} = \left\{ (X, Y) \mid (X, Y) = \delta(X', Y'), (X', Y') \in T_{FDH}, \delta \geq 0 \right\},$$
$$T_{NIRS} = \left\{ (X, Y) \mid (X, Y) = \delta(X', Y'), (X', Y') \in T_{FDH}, 0 \leq \delta \leq 1 \right\},$$
$$T_{NDRS} = \left\{ (X, Y) \mid (X, Y) = \delta(X', Y'), (X', Y') \in T_{FDH}, \delta \geq 1 \right\}. \quad (2)$$

In order to evaluate RTS in FDH technology the reference technology methods require the evaluation of the input or output radial efficiency scores of a given unit $(X_o, Y_o)$ in technology $T_{FDH}$ and/or in some of three technologies (2). For the sake of being definite, we limit our investigation to the case of input radial efficiency.

The input radial efficiency score $E^i_{FDH}(X_o, Y_o)$ in technology $T_{FDH}$ can be evaluated by solving
The next mixed integer linear problem:

$$E_{FDH}^i = \min \theta$$

subject to

$$\sum_{j=1}^{n} X_j \lambda_j \leq \theta X_o,$$

$$\sum_{j=1}^{n} Y_j \lambda_j \geq Y_o,$$

$$\sum_{j=1}^{n} \lambda_j = 1,$$

$$\lambda_j \in \{0, 1\}, \; j = 1, \ldots, n,$$

(3)

The evaluation of the input radial efficiency of unit \((X_o, Y_o)\) in any of the three associated technologies \([2]\) requires more computational effort. In the paper \([7]\), the following general model of the mixed integer nonlinear problems was developed for this purpose:

$$\theta_o = \min \theta$$

subject to

$$\sum_{j=1}^{n} X_j \lambda_j \leq \theta X_o,$$

$$\sum_{j=1}^{n} Y_j \lambda_j \geq Y_o,$$

$$\lambda_j = \delta w_j,$$

$$w_j \in \{0, 1\}, \; j = 1, \ldots, n,$$

$$\delta \in \Gamma, \sum_{j=1}^{n} w_j = 1,$$

(4)

where the set \(\Gamma\) is defined different by depending on the associated technology, in which the evaluation takes place. Actually, for the CRS, NIRS and NDRS technologies, we use

\[
\Gamma_{CRS} = \{ \delta \mid \delta \geq 0 \}, \quad \Gamma_{NIRS} = \{ \delta \mid 0 \leq \delta \leq 1 \}, \quad \Gamma_{NDRS} = \{ \delta \mid \delta \geq 1 \}.
\]

(5)

Let \(E_{CRS}^i(X_o, Y_o), E_{NIRS}^i(X_o, Y_o)\) and \(E_{NDRS}^i(X_o, Y_o)\) be the input radial efficiency scores of unit \((X_o, Y_o)\) in the reference technologies, in other words, equal to the optimal value \(\theta_o\) of problem \([4]\) with the appropriately specified set \(\Gamma\).

Next, we consider the notion of global RTS (GRS) in the case of FDH technology.

**Definition 1** (Banker \([2]\)). Unit \((X_o, Y_o)\) is at the most productive scale size (MPSS) if for all units \((\varphi X_o, \psi Y_o) \in T_{FDH}\), where \(\varphi, \psi > 0\), the ratio \(\psi/\varphi \leq 1\).

For verifying whether unit \((X_o, Y_o)\) is at MPSS it is necessary to solve the following optimization
According to Definition 1, unit \((X_0, Y_0)\) is at MPSS if and only if \(M^* = 1\).

**Definition 2.** Unit \((\varphi^* X_0, \psi^* Y_0)\) \(\in T_{FDH}\) is called a scale reference unit (SRU) of unit \((X_0, Y_0)\) if \(\varphi = \varphi^*, \psi = \psi^*\) is an optimal solution to problem (6), that is \(\psi^*/\varphi^* = M^*\).

It is worthy of note that unit \((X_0, Y_0)\) may have more than one SRU.

**Theorem 1** (Podinovski [16]). Assume that unit \((X_0, Y_0)\) is efficient in technology \(T_{FDH}\) and is not at MPSS. Let \((\varphi X_0, \psi Y_0)\) \(\in T_{FDH}\) be its SRU. Then either both \(\varphi\) and \(\psi\) are greater than 1 or both are smaller than 1.

It follows from this theorem, that we can classify the SRUs of the efficient unit \((X_0, Y_0)\) into those smaller \((\varphi < 1\) and \(\psi < 1\)\) and those larger \((\varphi > 1\) and \(\psi > 1\)\) than unit \((X_0, Y_0)\).

The definition below categorizes all efficient units into the four mutually exclusive types of global returns to scale (GRS) [16, 18].

**Definition 3.** Suppose that unit \((X_0, Y_0)\) is efficient in technology \(T_{FDH}\). Then unit \((X_0, Y_0)\) displays
1. global constant RTS (G-CRS) if unit \((X_0, Y_0)\) is at MPSS,
2. global decreasing RTS (G-DRS) if all its SRUs are smaller than unit \((X_0, Y_0)\),
3. global increasing RTS (G-IRS) if all its SRUs are larger than unit \((X_0, Y_0)\),
4. global sub-constant RTS (G-SCRS) if some its SRUs are smaller, and some are larger than unit \((X_0, Y_0)\), but unit \((X_0, Y_0)\) itself is not at MPSS.

The following theorem [16] presents us a practical tool for the GRS characterization of efficient units in the FDH model. It is based on the evaluation of input efficiency of unit \((X_0, Y_0)\) in FDH model and its NIRS and NDRS reference technologies.

**Theorem 2** (Podinovski [16]). Assume that unit \((X_0, Y_0)\) is efficient in technology \(T_{FDH}\), then it displays
1. G-CRS if and only if \(E_{NDRS}^i(X_0, Y_0) = E_{NIRS}^i(X_0, Y_0) = E_{FDH}^i(X_0, Y_0)\);
2. G-DRS if and only if \(E_{NDRS}^i(X_0, Y_0) < E_{NIRS}^i(X_0, Y_0) \leq E_{FDH}^i(X_0, Y_0)\);
3. G-IRS if and only if \(E_{NIRS}^i(X_0, Y_0) < E_{NDRS}^i(X_0, Y_0) \leq E_{FDH}^i(X_0, Y_0)\);
4. G-SCRS if and only if \(E_{NDRS}^i(X_0, Y_0) = E_{NIRS}^i(X_0, Y_0) < E_{FDH}^i(X_0, Y_0)\).

As it was remarked above, evaluating input radial efficiency of unit \((X_0, Y_0)\) in FDH model requires solving a mixed integer linear problem. On the other hand, evaluating its efficiency in the NIRS and NDRS technologies is the more complicated problem and requires solving mixed integer nonlinear problems.

Below we develop algorithms that allow us to construct sections on the frontier of the FDH model by two-dimensional planes, which allows us to identify the GRS type of an efficient unit \((X_0, Y_0)\) without computing its efficiency scores in reference technologies.
Algorithms for Construction of Efficient Frontier for NIRS and NDRS Technologies

In papers [9, 10], frontier visualization methods were proposed for FDH model based on enumeration and optimization algorithms. In this section, two methods are developed for construction of efficient frontier for NIRS and NDRS models. This approach allows us to evaluate returns to scale in the FDH model without additional solution of two non-linear mixed integer optimization problems for NIRS and NDRS reference technologies.

Consider any unit \((X_o, Y_o) \in T_{FDH}\) and any two nonzero vectors (directions) \(d_1, d_2 \in E^{m+r}\), such that \(d_1\) is not parallel to \(d_2\). Define two-dimensional plane in the space \(E^{m+r}\) as

\[
\Pi(X_o, Y_o, d_1, d_2) = (X_o, Y_o) + \alpha d_1 + \beta d_2, \tag{7}
\]

where \(\alpha\) and \(\beta\) are scalar parameters. The above plane passes through the point \((X_o, Y_o)\) and is spanned by the vectors \(d_1\) and \(d_2\).

Denote \(\text{WEff}_{P, \text{FDH}}\) the weakly (Pareto) efficient frontier of model FDH. Define the intersection of this frontier with the two-dimensional plane (7) as follows:

\[
\text{Sec}(X_o, Y_o) = \{(X, Y) \mid (X, Y) \in \Pi(X_o, Y_o, d_1, d_2) \cap \text{WEff}_{P, \text{FDH}}\}. \tag{8}
\]

Following papers [9, 12], and by taking different directions \(d_1\) and \(d_2\), one can construct various sections of the weakly efficient frontier \(\text{WEff}_{P, \text{FDH}}\). For our purpose (construction of sections of the frontier for NIRS and NDRS technologies) we should choose directions \(d_1 = (X_o, 0) \in E^{m+r}\) and \(d_2 = (0, Y_o) \in E^{m+r}\). Introducing parameters \(\varphi = 1 + \alpha\) and \(\psi = 1 + \beta\), we observe that the plane (7) consists of all points that can be written in the form \((X, Y) = (\varphi X_o, \psi Y_o)\). According to [12], we call such section \(\text{Sec}(X_o, Y_o)\) generalized production function. The corresponding sections \(\text{Sec}(X_o, Y_o)\) of the NIRS and NDRS technologies can be used to evaluate RTS of the FDH model at every point of the FDH section without additional solutions of nonlinear mixed integer optimization problems.

Figure 1 illustrates the work of Algorithm 1. It starts with vertexes \(Z_l = (z^\alpha_l, z^\beta_l), l = 1, \ldots, L\), generated by one of the algorithms described in [9, 10]. Next, Algorithm 1 finds \(k_0\), the index of maximal peak of the unit that exhibits maximal production scale size. In this case, \(k_0 = 9\), hence point \(P_1 = Z_9\) is one with a most productive scale size. Then, define a region \(D_1\) where Algorithm 1 finds next points \(P_2\) and \(P_2'\) of the two-dimensional section for NIRS technology. Iterations are repeated till the last point is found, in our case point (vertex) \(P_4\).

**Proposition 1.** Algorithm 1 constructs the generalized production function for NIRS technology for a given unit in a finite number of steps.

Figure 2 explains the main principle of Algorithm 2. As distinct from the previous algorithm, Algorithm 2 finds at first an index of minimal peak among points that exhibit maximal productive scale size. According to Fig. 2, this is point \(P_1(Z_7)\). Then the region \(D_1\) is determined, where points \(P_2\) and \(P_2'\) are found, and iterations are repeated. Hence, the algorithm moves from right to the left till the last point is found, in our case, this is point \(P_4\).

**Proposition 2.** Algorithm 2 constructs the generalized production function for NDRS frontier for a given unit in a finite number of steps.
Algorithm 1. Construction of production function for NIRS technology

**Input:** $Z_l = (z_\alpha^l, z_\beta^l)$, $l = 1, \ldots, L$, are vertexes of production function of unit $(X_o, Y_o)$ in FDH technology

**Output:** $S_l$, a production function of unit $(X_o, Y_o)$ in NIRS technology.

1: $P_0 \leftarrow (p_\alpha^0, p_\beta^0) = (0, 0)$, the first point of the section.
2: $D_0 \leftarrow \{1, \ldots, L\}$, the index set of the section in FDH model, it contains all vertexes of the two-dimensional section at the beginning.
3: $R_0 \leftarrow \{j \mid z_\beta^j \geq \frac{z_\beta^l}{z_\alpha^l}, l \in D_0\}$, index set of points with maximal productive scale size.
4: $k_0 \leftarrow \arg \max_{j \in R_0} \{z_\alpha^j\}$, the index of maximal peak of the unit that exhibits maximal production scale size.
5: $P_1 \leftarrow Z_{k_0}$, the second point of the NIRS production function.
6: $S \leftarrow [P_0, P_1]$, the first segment of the NIRS section.
7: $i \leftarrow 1$, initialize counter.
8: while $D_i \leftarrow \{l \mid z_\alpha^l > p_\alpha^i, z_\beta^l > p_\beta^i, l \in D_{i-1}\} \neq \emptyset$ do
9: $R_i \leftarrow \{j \mid z_\beta^j \geq \frac{z_\beta^l}{z_\alpha^l}, l \in D_i\}$, the index set of points with maximal productive scale size, that are contained in set $D_i$.
10: $k_i \leftarrow \arg \max_{j \in R_i} \{z_\alpha^j\}$, the index of maximal peak of unit that exhibits maximal productive scale size.
11: $P_{i+1} = (p_\alpha^i, p_\beta^i) \leftarrow Z_{k_i}$, the next angular point of a new section.
12: $P_{i+1}' \leftarrow \left(\frac{p_\alpha^i}{P_{i+1}}, \frac{p_\beta^i}{P_{i+1}}\right)$, a new angular point between points $P_i$ and $P_{i+1}$.
13: $S \leftarrow S \cup [P_i, P_{i+1}'] \cup [P_{i+1}', P_{i+1}]$, add two new segments to the constructing section.
14: $i \leftarrow i + 1$, increment counter.
15: end while
16: $S \leftarrow S \cup \{P_i + \gamma e_{\alpha} \mid \gamma \geq 0\}$, add a horizontal ray going out from the last point $P_i$ of the section.
17: Output $S$.

Computational Experiments

The algorithms described in this paper have been tested with the use of real-life data sets from different areas: Russian banks, Norwegian municipal nursing- and home-care services and Swedish electricity utilities. Original data sets are described in papers [5, 6, 8]. Computational results are illustrated in this paper on the data from Swedish electricity utilities. The number of production units in this dataset is 163. The model has four inputs and four outputs. On the input side we use kilometers of low- and high-voltage power lines and total transformer capacity in kVA as the capital variables. Labor is measured in full-time equivalent employees. As the four outputs, we used the total amount of low- and high-voltage electricity in MWhs delivered to customers and the number of low- and high-voltage customers served. All computational experiments were conducted using a personal computer with Intel Core i3 CPU operating at 3.33 GHz.

In our computational experiments, constructions of stepwise production functions with the help of enumeration algorithm [10] for all units in the FDH model took 0.0893 s. At the same time, constructions of production functions and sections of NIRS model for all units took 0.0965 s. And constructions of production functions and sections of NDRS models together for all units...
These figures showed that the time needed for section constructions for NIRS and NDRS models is negligibly small in comparison with a time required for constructions of production functions for FDH model.

Figures 3 and 4 present sections of the FDH frontier obtained in the experiments with dataset described above, the sections are constructed for unit 290. Figure 3 shows the whole two-dimensional section of the efficient frontier going through unit 290. Figure 4 represents an en-
Algorithm 2. Construction of production function for NDRS technology

Input: \( Z_l = (z_\alpha^l, z_\beta^l), l = 1, \ldots, L \), are vertexes of production function of unit \((X_o, Y_o)\) in FDH model

Output: \( S \), a production function of unit \((X_o, Y_o)\) in NDRS technology.

1: \( D_0 \leftarrow \{1, \ldots, L\} \), the index set of the section in FDH model, it contains all vertexes of the twodimensional section at the beginning.

2: \( R_0 \leftarrow \left\{ j \mid \frac{z_j^\beta}{z_j^\alpha} \geq \frac{z_l^\beta}{z_l^\alpha}, l \in D_0 \right\} \), index set of points with maximal productive scale size.

3: \( k_0 \leftarrow \arg \min_{j \in R_0} \{z_j^\alpha\} \), the index of minimal peak of the unit that exhibits maximal productive scale size.

4: \( P_1 \leftarrow Z_{k_0} \), the first point of the NDRS section.

5: \( S \leftarrow \{\gamma P_1 \mid \gamma \geq 1\} \), ray is added going out radially from point \( P_1 \) to infinity.

6: \( i \leftarrow 1 \), initialize counter.

7: \( \textbf{while } D_i \leftarrow \{l \mid z_l^\alpha < p_i^\alpha, z_l^\beta < p_i^\beta, l \in D_{i-1}\} \neq \emptyset \textbf{ do } \)

8: \( R_i \leftarrow \left\{ j \mid \frac{z_j^\beta}{z_j^\alpha} \geq \frac{z_l^\beta}{z_l^\alpha}, l \in D_i \right\} \), the index set of points with maximal productive scale size, that are contained in set \( D_i \).

9: \( k_i \leftarrow \arg \min_{j \in R_i} \{z_j^\alpha\} \), the index of minimal peak of unit that exhibits maximal productive scale size among units belonging to \( R_i \).

10: \( P_{i+1} = \left( p_i^\alpha, p_i^\beta \right) \leftarrow Z_{k_i} \), the next angular point of a new section.

11: \( P'_{i+1} \leftarrow \left( p_i^\alpha, p_i^\beta + \frac{p_i^\alpha}{P_i^\alpha + P_{i+1}^\alpha} \right) \), a new angular point between points \( P_i \) and \( P_{i+1} \).

12: \( S \leftarrow S \cup \{P_i, P'_{i+1}\} \cup [P_{i+1}, P_{i+1}], \) add two new segments to the section \( S \). Observe, that points of the section are added in the reverse order, that is point \( P_{i+1} \) is situated on the left from point \( P_i \).

13: \( i \leftarrow i + 1 \), increment counter.

14: \( \textbf{end while} \)

15: \( S \leftarrow S \cup \{P_i - \gamma e_\beta \mid \gamma \geq 0\} \), add to the section a vertical ray starting from last point \( P_i \) and going down.

16: Output \( S \).

larged part of the section in the vicinity of the origin. The coordinates of this unit in the figure are \((1, 1)\). The red line corresponds to FDH-NDRS technology, the blue line associates with FDH-NIRS technology, the gray dashed line is the part of the efficient frontier of FDH-VRS model. If lines FDH-NDRS, FDH-NIRS, and FDH-VRS are constructed, then RTS can be easily evaluated at any angular point \( Z_1, \ldots, Z_{15}, \ldots, Z_{35} \) on the curve. Take, for example, point \( Z_8 \). This point exhibits increasing RTS, since, according to the results of this paper, this point belongs to the NDRS line. Point \( Z_{12} \) exhibits constant RTS since it belongs to both NDRS and NIRS lines. Point \( Z_{15} \) is situated very close to the line NIRS, so it exhibits decreasing RTS.

Conclusions

Our work develops an approach for visualization of FDH frontier with increasing and decreasing returns to scale. Our method requires much fewer computations then existing ones. Our computational experiments documented this fact. Next, if the NDRS and NIRS section are constructed, it is not necessary to solve two additional mixed integer optimization problems in order to evaluate
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References


