AK-P: An Active Learning Method Combining Kriging and Probability Density Function for Reliability Analysis

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Abstract. An important challenge in structural reliability is to reduce the number of calls to the performance function. To reduce computational burden, surrogate models are commonly used. Kriging, one of the meta-models, is widely used as a surrogate for the original model in structural reliability analysis. In this paper, an active learning method combining Kriging and probability density function is proposed to improve the computational efficiency of AK-MCS. The proposed method, in general, provides a more efficient way by selecting the next best point effectively and adding it to the design of experiments to update the surrogate model more accurately. One example is used to demonstrate the efficiency of the proposed method.

Introduction

It is important to assess the probability of failure for a given mechanical structure considering the randomness of input variables. Generally, the probability of failure is defined as

\[ P_f = \int_{G(x) \leq 0} f(x) \, dx, \tag{1} \]

where \( f(x) \) is the joint probability density function (PDF) of input variables \( X \), and \( G(\cdot) \) is the performance function of analyzed mechanical structure. The variable space is divided by the limit state \( G(x) = 0 \) into two domains, i.e., the failure domain \( G(x) \leq 0 \) and the safe domain \( G(x) > 0 \).

The probability of failure in Eq.1 can also be described as follows:

\[ P_f = \int I_{G(x) \leq 0} f(x) \, dx, \tag{2} \]

where \( I_{G \leq 0} \) is a failure indicator function that is defined as:

\[ I_{G(x)} = \begin{cases} 1 & G(x) \leq 0 \\ 0 & G(x) > 0 \end{cases}. \tag{3} \]

\[ \hat{P}_f = \frac{n_{G(x) \leq 0}}{N_{MC}}. \tag{4} \]

In engineering problems, the performance function \( G(x) \) of the target structure is usually very complex and often impossible to get its explicit formulation. Therefore, the probability of failure \( P_f \) cannot be assessed directly by multi-dimensional \( f(x) \) integrating over the entire failure domain in Eq.1. To solve the integral effectively, many methods have been proposed. Monte Carlo Simulation (MCS) [1, 2] is a widely used and robust methodology which provides very accurate results that can be viewed as references of other new methods. \( \hat{P}_f \), an estimation of failure probability, is defined as Eq.4, where \( N_{MC} \) is the population of all input random variables and \( n_{G(x) \leq 0} \) is the population of
the samples located in the failure domain \((G(\mathbf{x}) \leq 0)\). For small failure probability events, \(N_{MC}\) should be large enough in order to get a convergent result, i.e., \(N_{MC} = (10^2 \sim 10^4)/P_f\) [3]. Problems analyzed in engineering about structural reliability always have very small probability, and \(N_{MC}\) should be large enough. Especially, the performance function is usually implicit and the finite element model (FEM) is utilized to get the response, which is time-consuming.

To reduce the total calls to the performance function and relieve the pressure of computation, some variance reduction techniques were proposed, e.g., subset simulation (SS) [4-6], important sampling (IS) [7, 8], and line sampling (LS) [9]. SS computes the failure probability as a product of a series of conditional probabilities estimated easily by Markov Chain Monte Carlo (MCMC) simulation [10]. The basic principle of IS is to generate points around the most probable point (MPP) that reside around the vicinity of the limit state surface. Although IS is one of the most effective variance reduction techniques, its success depends on the prior knowledge about the failure domain and the auxiliary distribution assignment [11]. LS computes the probability of failure based on the optimal important direction which is from the origin of coordinate to MPP in the standard normal space [9]. These variance reduction techniques effectively improve the efficiency of computation through a reduced number of calls to the performance function compared to MCS. However, these variance reduction technologies are difficult when applied to the engineering structural reliability problems with FEM models.

Constructing a model to substitute the complex performance function has been a key research thrust. In recent years, many machine learning methods are proposed to construct the substituted model in structural reliability problems, e.g., neural networks [12-14], support vector machines (SVM) [15, 16]. These machine learning methods are more suitable to matching performance function with highly nonlinear input-output relationships [17]. Kriging, one of the machine learning methods, is widely used as a surrogate for the original model in structural reliability analysis [3, 8, 18-22], which presents interesting characteristics such as exact interpolation and a local index of uncertainty on the prediction. Therefore, the Kriging model is employed combing with probability density function in this research.

The paper is organized as follows. Section II reviews the basic theory of Kriging method. Section III presents AK-P: a new active learning method combing Kriging and probability density function. Section IV presents the implementation progress of reliability analysis by the proposed active learning method AK-P. One example is employed in section V to illustrate the efficiency of the proposed method. Section VI is the conclusion.

**Basic Theories of Kriging for Structural Reliability Analysis**

Kriging developed the Kriging approach for geo-statistics, and Matheron refined this surrogate model [23]. The Kriging model includes two parts: a linear regression model and a stochastic process [24]:

\[
\hat{G}(x) = f^T(x)\beta + Z(x),
\]

where \(F(x, \beta)\) is the deterministic part and \(Z(x)\) is a stationary Gaussian process with zero mean, and the covariance between two points can be defined as

\[
\text{Cov}(Z(x_i), Z(x_j)) = \sigma^2 R(x_i, x_j), \quad i, j = 1, 2, ..., N
\]

where \(\sigma^2\) is the process variance and \(R(x_i, x_j)\) is the Gaussian correlation function. Irfan Kaymaz [25] mentions Gaussian correlation function is more flexible and suitable for the performance function with nonlinearity which usually occurs in engineering, so it is adopted in this paper.
A New Active Learning Method

Learning Function

The learning function is used to find the next best point in the random space including learning criterion and stop condition, and then the point will be added to the design of experiments (DoE) to update the Kriging model more accurately.

EFF (Expected feasibility function) is proposed by Bichon [26], and is defined as

\[
\text{EFF}(x) = \int_{\mathcal{G}} \left[ \varepsilon - \tilde{G}(x) - \tilde{G}(x) \right] f_{\tilde{G}}(\tilde{G}) d\tilde{G},
\]

where \( \tilde{G}(x) \) is the constraint equation and \( f_{\tilde{G}} \) is the PDF of \( \tilde{G}(x) \). \( \varepsilon = k\sigma_{\tilde{G}}(x) \), and \( k \) is always equal to 2. To guarantee the accuracy near the limit state function, \( G^- = \tilde{G}(x) - \varepsilon \), \( G^+ = \tilde{G}(x) + \varepsilon \).

In reliability analysis, \( \tilde{G}(x) = 0 \).

However, this learning function approximates the whole limit state which causes a slow convergence.

Table 1. Definition of the learning criterion and the stopping condition for the learning functions EFF and U.

<table>
<thead>
<tr>
<th>Learning function</th>
<th>EFF</th>
<th>U</th>
</tr>
</thead>
<tbody>
<tr>
<td>Learning criterion</td>
<td>( \max(\text{EFF}(x)) )</td>
<td>( \min(\text{U}(x)) )</td>
</tr>
<tr>
<td>Stop condition</td>
<td>( \max(\text{EFF}(x)) \leq 0.001 )</td>
<td>( \min(\text{U}(x)) \geq 2 )</td>
</tr>
</tbody>
</table>

The learning function U is proposed by Echard [18] as,

\[
|\tilde{G}(x)| - U(x)\sigma_{\tilde{G}}(x) = 0,
\]

According to Eq.8, higher weight is applied to the points in the close neighborhood of the predicted limit state rather than further ones with high Kriging variance like EFF can do.

Table 1 sums up the learning parameters for EFF and U [18].

Principles of the Proposed Methodology

In this paper, a new learning function named AK-P is proposed to combining Kriging and probability density function for reliability analysis.

If the U function value is low, the sample point corresponding to the U function shows three characteristics: 1. The sample point is close to the limit state. 2. The sample point with a higher prediction variance has an important uncertainty than others. 3. The sample points has the characteristics presented in 1 and 2 that is shown in Fig. 1 (S and F stand for safe region and failure region, respectively).

If these points which have a low probability density as well as a small value of U that are added to the DoE, the number of calls to the performance function will increase, while the accuracy of final estimation of failure probability has no obvious improvement. To improve the efficiency of reliability analysis, the points with the probability density function and the value of U are not considered.

In this paper, a new learning function combing the probability density function of the sample points and the U function to select the next best point to the DoE is proposed to update the Kriging...
model in a mojire effective way. The sample points selected not only have a low U value, but also have a relative large probability density. The new learning function $U^*(X)$ is defined as:

$$U^* = \min_{L} \frac{U(x_L)}{f(x_L)}, L = 1, 2, \ldots, N,$$

where $U(X)$ is obtained by AK-MCS, and $f(X)$ is the joint probability density function of the input random variables. The inverse function of the PDF of the input random variables shows that the larger the PDF, the lower the learning function. The point with the minimum $U^*$ value in the random variables space will be selected.

The learning method based on AK-MCS for the reliability problems can be performed as Fig. 2:

**Procedure of the Propose Method**

The details of proposed methodology can be explained as follows:

a) Construction of the initial Kriging model: Generate $N_1$ samples according to the distribution of variables, and select randomly $N_0$ points in $N_1$ to construct an initial Kriging model.

b) Computation of the Kriging prediction of all candidate sample points and find the next best point: Compute the predictions of all sample points utilizing the Kriging model constructed in Step a. Find the next best point according to the proposed method described in Section 3.

c) Convergence criterion judgment: Estimate the value of learning function $U$ and determine whether the convergence criterion is met. If criterion is not met, add the next best point and its response to the DoE, and go to Step b to improve the accuracy of the Kriging model. If else, go to next step.

d) Estimation of the probability of failure: Compute the probability of failure based on the predictions obtained in Step b using MCS method. If the coefficient of variation is low enough, the proposed method stops, and the last estimation of failure probability is considered as the result of the reliability analysis. If not, go to Step a and start the process again.

**Numerical Examples**

One example is compared in this section, and the program is run by MATLAB 2014b that is
installed on a computer with an Intel Core i7-6700, a 8.00 GB RAM, and an operation system of Windows 10 enterprise edition.

A Highly Nonlinear Two-Dimensional Example

A highly nonlinear performance function [27] is defined by

\[
G(x) = \sin \left( \frac{5x_1}{2} + 2 - \frac{(x_1^2 + 4)(x_2 - 1)}{20} \right),
\]

where \(x_1\) and \(x_2\) are two independently and normally distributed random variables with \(x_1 \sim N(1.5, 1)\) and \(x_2 \sim N(2.5, 1)\).

The comparisons of AK-MCS, MCS and proposed method AK-P are given in Table 2. The probability of failure obtained by MCS is used as the reference. To assess the accuracy of AK-MCS based on the learning function \(U\) and the proposed method, 12 sample points are randomly selected by MCS as the initial DoE for the Kriging process, and \(2 \times 10^5\) points are generated randomly according to the distributions of the random variables as the candidate sample points for the selection of next best point.

\(T\) is the CPU time of the program measured in seconds. The smaller the \(T\), the higher the efficiency. The probability of failure obtained by AK-P is almost the same as the results from the AK-MCS, while the computational efficiency is higher than AK-MCS in this paper. Some details are shown in Fig. 3 and Fig. 4, respectively. LSF is short for limit state function. Fig. 4 illustrates the proposed method puts more weight in the region with a higher probability density in reliability analysis compared to Fig. 3. As shown in Fig. 3 and Fig. 4, the sample points added to the DoE in the method AK-P are centered more on the region with a large value of PDF, while the points are relatively scattered in the method AK-MCS.

![Figure 3](image1.png) ![Figure 4](image2.png)

**Table 2. Reliability results of example 1.**

<table>
<thead>
<tr>
<th>Method</th>
<th>(N_{\text{call}})</th>
<th>(P_f)</th>
<th>(T) (s)</th>
<th>(\varepsilon_{P_f}) (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>MCS</td>
<td>2e5</td>
<td>0.03172</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>AK-MCS</td>
<td>39</td>
<td>0.03160</td>
<td>37.366</td>
<td>0.378</td>
</tr>
<tr>
<td>AK-P</td>
<td>21</td>
<td>0.03178</td>
<td>12.407</td>
<td>0.189</td>
</tr>
</tbody>
</table>

**Conclusion**

In this paper, a new learning method based on probability density function and learning function \(U\) is proposed for reliability analysis. One example, a highly nonlinear two-dimensional example is used to demonstrate the proposed method. According to the numerical example, we know that the
The proposed method is generally more accurate and efficient than the AK-MCS with learning function $U$. However, it does not mean that the proposed method is more efficient than AK-MCS for all cases. The main reasons is that the updated Kriging model selects the next best point avoid the area with small probability using the proposed learning function. The proposed method can be a useful tool for reliability analysis, especially for problems with complex implicit limit state functions.

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References


