Modeling and Analysis of Installation Error of Uniaxial Gyro North Finder

Yang-yang CHU*, Mei-ping WU and Ju-liang CAO

College of Intelligent Science and Engineering, National University of Defense Technology, ChangSha 410073, China

*Corresponding author

Keywords: Single gyro, North finder, Installation error, Error modeling, Error analysis.

Abstract. Single gyro north finder is widely used at present because of its low price and convenience. The installation error is modeled and analyzed for it in this paper. Firstly, the single-gyro north-seeking principle is introduced. Then, the trajectory of the gyro sensitive axis when there is installation error of the north finder is analyzed comprehensively, and a unified model of the angular velocity which is measured by gyroscope is established. Based on the model, error analysis is performed for the commonly used static multi-position north seeking methods, which provides a reference for the design and maintenance of the north finder.

Introduction

In the practice of human production and life, geographical orientation is indispensable information. Single gyro north finder calculates true north by measuring the component of the earth's angular velocity in different directions. At present, the static methods such as two-position and four-position are commonly used. It is widely applied in weapon targeting, rocket launching, tunnel construction and other military and civil applications [1]. Compared with gyro-based theodolite, the strapdown north finder has the advantages of low price, small size, fast speed, and simple maintenance and convenience. The north finder is an independent position indication device that is attached to the carrier. There are many factors that cause the error of the obtained azimuth. This article mainly analyzes the error caused by installation error.

Principle of the North Finder

The strapdown north seeker measures the component of the earth's rotational angular velocity to calculate the carrier’s azimuth. The earth's rotational angular velocity vector $\omega_{ie}$ is parallel to the earth's axis. At any latitude $L$ on the earth, the angular velocity can be decomposed into a horizontal component along the meridian direction $\omega_N=\omega_{ie} \cos L$ and a vertical component $\omega_U=\omega_{ie} \sin L$, as shown in Fig. 1. When the gyro sensitive axis is in the local horizontal plane and the azimuth is $\phi$, its sensitive angular velocity is

$$\omega = \omega_{ie} \cos L \cos \phi. \quad (1)$$

And the gyro output is [2]:

$$D = D_0 + K\omega_{ie} \cos L \cos \phi + \epsilon. \quad (2)$$

Where $D_0$ is the gyro output that corresponds to the zero bias, $K$ is the gyro scale factor, and $\epsilon$ is the random error.
The two-position north seeking usually selects two locations with a difference of 180°, which can eliminate the error caused by the zero bias. The gyro outputs of the two locations respectively are:

\[ D_1 = D_0 + K \omega_{ie} \cos L \cos \varphi + \varepsilon_1 \]
\[ D_2 = D_0 - K \omega_{ie} \cos L \cos \varphi + \varepsilon_2 \]  

Ignoring the change of the gyro bias and random error, calculating the azimuth:

\[ \varphi = \arccos \left( \frac{D_1 - D_2}{2K \omega_{ie} \cos L} \right) \]  

The four-position north seeking generally selects four locations with a difference of 90° to perform north seeking. This method does not rely on the local geographic latitude and the scale factor of the gyro, but the time of seeking north is longer than that of the two-position method. The gyro outputs of the four locations respectively are:

\[ D_1 = D_0 + K \omega_{ie} \cos L \cos \varphi + \varepsilon_1 \]
\[ D_2 = D_0 - K \omega_{ie} \cos L \sin \varphi + \varepsilon_2 \]
\[ D_3 = D_0 - K \omega_{ie} \cos L \cos \varphi + \varepsilon_3 \]
\[ D_4 = D_0 + K \omega_{ie} \cos L \sin \varphi + \varepsilon_4 \]  

Neglecting the change of the gyro bias and random error, solving the azimuth:

\[ \varphi = \arctan \left( \frac{D_4 - D_2}{D_1 - D_3} \right) \]  

Modeling of the Angular Velocity Obtained by the North Finder with Installation Error

Transform Relationships between Coordinate Systems. The north finder usually installs the gyroscope to the shaft, and the shaft drives the gyroscope to different positions for data collection. The geographic coordinate system \( OX_nY_nZ_n \) adopted in this paper is east, north, and the local vertical (up), and north by east is considered as positive. The axes of the coordinate system \( OX_bY_bZ_b \) are in the direction of the carrier’s lateral, longitudinal, and vertical vectors respectively. \( \varphi, \theta, \gamma \) are the yaw angle, pitch angle, and roll angle respectively. Where the gyro sensitive axis lies is the \( Y_g \) of the gyro coordinate system \( OX_gY_gZ_g \), and the rotation axis is the \( Z_s \) of the rotational axis coordinate system \( OX_sY_sZ_s \) [3]. This article discusses north finder working under horizontal conditions.

Because the north finder works under horizontal conditions, \( \theta \) and \( \gamma \) are small angles, the direction cosine matrix from the n-frame to the b-frame is:

\[
C_n^b = \begin{bmatrix}
\cos \varphi & -\sin \varphi & -\gamma \\
\sin \varphi & \cos \varphi & \theta \\
\gamma \cos \varphi & -\theta \sin \varphi & -\gamma \sin \varphi - \theta \cos \varphi & 1
\end{bmatrix}
\]  

The rotation shaft determines the direction of the \( Z_s \), therefore the shaft installation error only exists on the x-axis and y-axis with respect to the carrier and are denoted as \( \eta_x \) and \( \eta_y \). When they are small angles, the direction cosine matrix from the b-frame to the s-frame is:
The gyro sensitive axis determines the direction of the $\mathbf{Y}_g$, therefore the gyro installation error only exists on the x-axis and z-axis with respect to the shaft and are denoted as $\xi_x$ and $\xi_z$. When they are small angles, the direction cosine matrix from the s-frame to the g-frame is:

$$
C_{s}^{g} = 
\begin{bmatrix}
1 & -\xi_z & 0 \\
\xi_z & 1 & \xi_x \\
0 & -\xi_x & 1
\end{bmatrix}
$$

(9)

The shaft drives the gyro to different positions. When the system rotates clockwise $\alpha$ around z-axis, the direction cosine matrix from the so-frame to the si-frame is:

$$
C_{s0}^{si} = 
\begin{bmatrix}
\cos\alpha & -\sin\alpha & 0 \\
\sin\alpha & \cos\alpha & 0 \\
0 & 0 & 1
\end{bmatrix}
$$

(10)

**Analysis of Gyro Sensitive Axis Trajectory.** When the north finder works under ideal conditions and there is no installation error, the shaft is perpendicular to the local horizontal plane and the gyro sensitive axis is parallel to the horizontal plane. The trajectory formed by the gyro sensitive axis in one revolution is a standard circle parallel to the horizontal plane. However, deviation will occur when there is installation error [4].

When there is only the gyro installation error, the installation error of the gyro around the z-axis, $\xi_z$, is the system error and can be eliminated by the previous calibration. When there is an error around the x-axis, $\xi_x$, the surface formed by gyro sensitive axis in one revolution becomes a conical surface, as shown in Fig. 2. The angle between gyro sensitive axis and the horizontal plane is always $\xi_x$.

![Figure 2. The trajectory when there is gyro installation error.](image)

When there is only the shaft installation error, the shaft installation error is divided into $\eta_x$ and $\eta_y$ around the x-axis and y-axis. Fig. 3 shows the trajectory formed by gyro sensitive axis with only $\eta_x$ or $\eta_y$. It is a slope circle that intersects the horizontal plane. At each point, the angle between the gyro sensitive axis and the horizontal plane is different.

![Figure 3. The trajectory when $\eta_x$ or $\eta_y$ is present.](image)
If the gyro and the shaft have error at the same time, the situation is even more complicated. Fig. 4 shows the trajectory when $\xi_x$ and $\eta_x$ are present at the same time. The gyro sensitive axis forms a conical surface when it rotates one revolution as shown in the figure.

![Figure 4. The trajectory when $\xi_x$ and $\eta_x$ are present.](image)

**Installation Error Modeling.** The azimuth is obtained through the outputs of the gyro. When there is installation error, it is necessary to analyze the actual angular velocity of each position in order to calculate the impact on the accuracy of the north seeking. Therefore, it is necessary to establish a mathematical model description of the angular velocity that gyro measures at each position.

Because it is the shaft that drives the gyro, when analyzing the installation error, it cannot simply multiply the rotation matrix finally to obtain the measurement angular velocity of the next position. The gyro measurement angular velocity should be obtained through the rotational axis coordinate system of different positions.

$$\omega_{ie} = C^b_{si} C^{s0}_{b} C^{0}_{n} \omega_{ie}$$  \hspace{1cm} (11)

Substituting Eq. 7-10 into Eq. 11, and $\omega_{ie} = [0 \quad \omega_{ie} \cos L \quad \omega_{ie} \sin L]^T$. Since only the angular velocity on the gyro sensitive axis is required, we only calculate the value on the y-axis. Ignoring the second-order small term, and the angular velocity of each position measured by gyro is deduced.

$$\omega = \omega_{ie} \cos L \left[ \cos \alpha (\xi_z \sin \varphi + \cos \varphi) + \sin \alpha (-\sin \varphi - \xi_z \cos \varphi) \right]$$

$$+ \omega_{ie} \sin L \left[ \cos \alpha (\theta + \eta_x + \xi_x) \right]$$  \hspace{1cm} (12)

**Error Analysis**

The error analysis is carried out for the single-position, two-position and four-position north seeking schemes. According to the gyro-measured angular velocity model above, the angular velocity at $0^\circ$, $90^\circ$, $180^\circ$, and $270^\circ$ respectively are:

$$\omega_{0^\circ} = \omega_{ie} \cos L (\xi_z \cos \varphi + \sin \varphi) + \omega_{ie} \sin L (\theta + \eta_x + \xi_x)$$

$$\omega_{90^\circ} = \omega_{ie} \cos L (\sin \varphi + \xi_z \cos \varphi) + \omega_{ie} \sin L (-\gamma - \eta_y + \xi_x)$$

$$\omega_{180^\circ} = \omega_{ie} \cos L (\xi_z \sin \varphi - \cos \varphi) + \omega_{ie} \sin L (-\theta - \eta_x + \xi_x)$$

$$\omega_{270^\circ} = \omega_{ie} \cos L (\sin \varphi + \xi_z \cos \varphi) + \omega_{ie} \sin L (\gamma + \eta_y + \xi_x)$$  \hspace{1cm} (13)

Since only the influence of the installation error is analyzed, it is assumed that the gyro measures the angular velocity without error. The calculated azimuth is thought as the true azimuth plus the small angle error, $\hat{\varphi} = \varphi + \Delta \varphi$.

**Single-position North Seeking.** According to Eq. 1, we have

$$\hat{\varphi} = \arccos \frac{\omega_{0^\circ}}{\omega_{ie} \cos L}$$  \hspace{1cm} (14)

$$\cos(\varphi + \Delta \varphi) = \cos \varphi - \Delta \varphi \sin \varphi = -\xi_z \sin \varphi + \cos \varphi + \tan L (\theta + \eta_x + \xi_x).$$  \hspace{1cm} (15)

The error solved is
\[ \Delta \phi = \xi_z - \tan \frac{L}{\sin \phi} (\theta + \eta_x + \xi_x). \]  

The north-seeking error is related to latitude and azimuth. The higher the latitude is, the greater the error is. When the azimuth is near the east or the west, \( \sin \phi \) is the largest and the error is the smallest. Error angles \( \theta \), \( \eta_x \), \( \xi_x \) have the same effect on the accuracy of north seeking. The Fig. 5 shows the north seeking error caused by an error angle of 20 arc-seconds.

![Figure 5. The north seeking error caused by an error angle of 20 arc-seconds.](image)

**Two-position North Seeking.** According to Eq. 4, we have

\[ \hat{\phi} = \arccos \left( \frac{60^\circ - 60^\circ}{2 \omega_0 \cos L} \right). \]  

\[ \cos (\phi + \Delta \phi) = \cos \phi - \Delta \phi \sin \phi = -\xi_z \sin \phi + \cos \phi + \tan L (\theta + \eta_x). \]  

The error solved is

\[ \Delta \phi = \xi_z - \tan \frac{L}{\sin \phi} (\theta + \eta_x). \]  

The north-seeking error is affected by \( \theta \) and \( \eta_x \), and related to latitude and azimuth. The influence coefficient is the same as the single-position. The error caused by gyro misalignment can be neglected when it is a small angle.

**Four-position North Seeking.** According to Eq. 6, we have

\[ \hat{\phi} = \arctan \left( \frac{90^\circ - 90^\circ}{60^\circ - 0^\circ} \right). \]  

\[ \tan \hat{\phi} = \frac{2 \omega_0 \cos L (\sin \phi + \xi_z \cos \phi) + 2 \omega_0 \sin L (\gamma + \eta_y)}{2 \omega_0 \cos L (-\xi_z \sin \phi + \cos \phi) + 2 \omega_0 \sin L (\theta + \eta_x)}. \]  

\[ \tan (\phi + \Delta \phi) = \frac{\tan \phi + \Delta \phi}{1 - \Delta \phi \tan \phi} = \frac{\cos L (\sin \phi + \xi_z \cos \phi) + \sin L (\gamma + \eta_y)}{\cos L (-\xi_z \sin \phi + \cos \phi) + \sin L (\theta + \eta_x)}. \]  

The error solved is

\[ \Delta \phi = \frac{\cos L (\sin \phi + \xi_z \cos \phi) + \sin L (\gamma + \eta_y)}{\tan \phi \cos L (\sin \phi + \xi_z \cos \phi) + \sin L (\theta + \eta_x)} - \tan \phi \left[ \cos L (-\xi_z \sin \phi + \cos \phi) + \sin L (\gamma + \eta_y) \right]. \]  

From the equation, it can be seen that \( \gamma \) and \( \eta_y \) have the same influence on the north-seeking error, and \( \theta \) and \( \eta_x \) have the same influence on the north-seeking error. When only the error angle \( \xi_z \) exists, we have

\[ \Delta \phi = \frac{\cos L (\sin \phi + \xi_z \cos \phi) - \tan \phi \cos L (-\xi_z \sin \phi + \cos \phi)}{\tan \phi \cos L (\sin \phi + \xi_z \cos \phi) + \cos L (-\xi_z \sin \phi + \cos \phi)} = \frac{\xi_z \cos \phi + \xi_z \tan \phi \sin \phi}{\tan \phi \sin \phi + \xi_z \sin \phi + \cos \phi} = \xi_z. \]  

\( \xi_z \) is the system error that can be eliminated by the previous calibration.

When only the error angle \( \gamma \) and \( \eta_y \) exist, we have

\[ \Delta \phi = \frac{\cos L (\sin \phi + \xi_z \cos \phi) - \tan \phi \cos L (\gamma + \eta_y)}{\tan \phi \cos L (\sin \phi + \xi_z \cos \phi) + \cos L (\gamma + \eta_y)} \]
\[
\Delta \phi = \frac{\cos L \sin \phi + \sin L \left( \gamma + \eta_y \right)}{\tan \phi \cos L \sin \phi + \sin L \left( \gamma + \eta_y \right)} - \tan \phi \cos L \cos \phi \frac{\sin L \left( \gamma + \eta_y \right)}{\tan \phi \sin L \left( \gamma + \eta_y \right)} + \cos L \cos \phi = \frac{\cos \phi}{\sin \phi + \frac{1}{\tan L \left( \gamma + \eta_y \right)}}. \tag{25}
\]

It is related to latitude and azimuth. The higher the latitude is, the greater the error is. When the azimuth is near the east or the west, \(\sin \phi\) is the largest, \(\cos \phi\) is the minimum, and the error caused is negligible. Fig. 6 shows the north seeking error caused by an error angle of 20 arc-seconds.

When only the error angle \(\theta\) and \(\eta_x\) exist, we have

\[
\Delta \phi = \frac{\cos L \sin \phi - \tan \phi \left[ \cos L \cos \phi + \sin L \left( \theta + \eta_x \right) \right]}{\tan \phi \cos L \sin \phi + \sin L \left( \theta + \eta_x \right)} = -\tan \phi \frac{\sin L \left( \theta + \eta_x \right)}{\cos L \cos \phi + \sin L \left( \theta + \eta_x \right)} = -\frac{\sin \phi}{\cos \phi + \frac{1}{\tan L \left( \theta + \eta_x \right)}}. \tag{26}
\]

It is related to latitude and azimuth. The higher the latitude is, the greater the error is, and error is the smallest when the azimuth is near the north or the south. Fig. 7 shows the north seeking error caused by an error angle of 20 arc-seconds.

**Conclusion**

There are many factors that cause the north-seeking error of the north finder. This paper establishes a gyro-measured angular velocity model and comprehensively analyzes north-seeking error caused by the installation error and tilting of the carrier.

Through the analysis, the following conclusions have been drawn: The installation error of the gyro around the z-axis, \(\xi_z\), always affects the accuracy of north seeking. It is a system error and can be eliminated by the previous calibration. In the single-position north seeking, the small angle error around the x-axis—base tilt angle, shaft installation error, and gyro installation error \(\theta\), \(\eta_x\) and \(\xi_x\) have the same effect on the accuracy of north-seeking, and the error is the smallest when the azimuth is near the east or the west. In the two-position north seeking, only \(\theta\) and \(\eta_x\) affect the accuracy of north seeking, and the gyro installation error \(\xi_x\) is negligible when it is a small angle. In the four-position
north seeking, the installation error around x-axis and y-axis both affect the north-seeking accuracy. When the initial azimuth is near the east or the west, the accuracy is mainly affected by $\theta$ and $\eta_x$. On the contrary, when the initial azimuth is near the south or the north, the accuracy is mainly affected by $\gamma$ and $\eta_y$.

It is concluded that the north finder working under horizontal conditions should be leveled as much as possible, and the installation error of the shaft should be usually checked.

References


