Multisplitting Iterative Methods with General Weighting Matrices for Solving Symmetric Positive Linear Complementarity Problem

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Abstract. In this paper, by making use of optimal models, we study the weighting matrices of the multisplitting parallel methods for solving the symmetric positive definite linear complementarity problem, which is a powerful alternative for solving the large sparse linear complementarity problems. In our multisplitting there is only one that is required to be P-regular splitting and all the others can be constructed arbitrarily, which not only decreases the difficulty of constructing the multisplitting of the coefficient matrix, but also relaxes the constraints to the weighting matrices (unlike the standard methods, they are not necessarily nonnegative diagonal scalar matrices or given in advance). Finally, we prove the convergence of this new method.

Introduction

This paper focuses on the linear complementarity problem, which is to find a pair of real vectors \( r \) and \( z \in \mathbb{R}^n \) such that

\[
r := q + Az \geq 0, z \geq 0, z^T (q + Az) = 0
\]  

(1)

Where \( A \in \mathbb{R}^{n \times n} \) and \( q \in \mathbb{R}^n \) are given real matrix and vector, respectively, and \( z^T \) denotes the transpose of the vector \( z \).

This problem has been intensely studied since the 1960s. Many alternative formulations have been investigated, and efficient algorithms have been proposed in Cottle et al. (1992)

[1]. Much attention has recently been paid on a class of iterative methods called the matrix-splitting method [2-4]. Matrix splitting method for linear complementarity problem exploits particular features of matrices such as the sparsity and the block structure. Such an approach is motivated by matrix splitting methods for the linear complementarity problem.

Recently, Wen presents a multisplitting parallel methods for solving the symmetric positive definite linear systems

[5]. This method is based on the matrix multisplitting technique introduced by O’Leary and White


In this paper, the multisplitting parallel methods for linear complementarity problem will be established, which is a generalization of multisplitting parallel methods for solving the symmetric positive definite linear systems [5]. Some sufficient conditions for convergence of the multisplitting parallel methods will be proposed, when the matrix \( A \) is an H-matrix with positive diagonal elements positive definite.

We shall use the following notation. A real \( n \times n \) matrix \( A = (a_{ij}) \) is called nonnegative (respectively, positive) and denoted \( A \geq 0 \) (respectively, \( A > 0 \)) if \( a_{ij} \geq 0 \) (respectively, \( a_{ij} > 0 \)) for \( 1 \leq i, j \leq n \). In accordance with this notation, if \( A, B \) are real \( n \times n \) matrices, we say that \( A \geq B \) (respectively, \( A > B \)) if \( A - B \geq 0 \) (respectively, \( A - B > 0 \)) and similarly for vectors. A real matrix \( A \) is called monotone if \( A \) is nonsingular and \( A^{-1} \geq 0 \). Let \( C \) be an \( n \times n \) matrix. By \( \text{diag}(C) \) we denote the \( n \times n \) diagonal matrix coinciding in its diagonal with \( C \).
Definition 1.1(7) A nonsingular matrix $A = (a_{ij}) \in \mathbb{R}^{n \times n}$ is called an $M$-matrix if $a_{ij} \leq 0$ for $i \neq j$ and $A^{-1} \geq 0$. The comparison matrix $< A > = (b_{ij}) \in \mathbb{R}^{n \times n}$ of a matrix $A = (a_{ij}) \in \mathbb{R}^{n \times n}$ is defined by $b_{ij} = \begin{cases} a_{ij}, & i = j \\ -a_{ij}, & i \neq j \end{cases}$, $i, j = 1,2,\cdots,n$. $A$ is said to be an $H$-matrix if its comparison matrix $< A >$ is an $M$-matrix.

Definition 1.2(2) Let $A$ be an $n \times n$ matrix. The representation $A = M - N$ is called a splitting of $A$ if $M$ is a nonsingular matrix. In addition, the splitting is (1) convergent if $\rho(M^{-1}N) < 1$; (2) regular if $M^{-1} \geq 0$ and $N \geq 0$.

Definition 1.3(9) Let $A$ be a nonsingular real $n \times n$ matrix, and suppose that for some $\alpha \in \mathbb{N}$, we are given matrices $M_i, N_i, E_i \in \mathbb{R}^{n \times n}, i = 1,2,\cdots, \alpha$, satisfying
1. $A = M_i - N_i$ for $i = 1,2,\cdots, \alpha$.
2. $M_i$ is nonsingular for $i = 1,2,\cdots, \alpha$.
3. $E_i$ is a diagonal matrix with nonnegative entries for $i = 1,2,\cdots, \alpha$, and $\sum_{i=1}^{\alpha} E_i = I$ (n×n identity matrix).

Then the collection of triples $(M_i, N_i, E_i), i = 1,2,\cdots, \alpha$ is called a multisplitting of $A$.

Definition 1.4(11) A matrix $A \in \mathbb{R}^{n \times n}$ is said to be a P-matrix if all its principal minors are positive. The class of such matrices is denote $P$.

The Multisplitting Iterative Method for the Linear Complementarity Problem

In order to achieve higher computing efficiency by making use of the information contained in the matrix $A$, based on the matrix splitting iteration method, we present a multisplitting iterative method for solving the linear complementarity problem (1).

Let the matrix $A$ is a symmetric positive definite matrix, $A = M_i - N_i (i = 1,2,\cdots,m)$ is the multisplitting of the $A, N = \{1,2,\cdots,n\}, N_i \subset N (i = 1,2,\cdots,m)$. Let

$$E_i = diag(\alpha_i^{(1)}, \alpha_i^{(2)}, \cdots, \alpha_i^{(n)}) \in \mathbb{R}^{n \times n} (i = 1,2,\cdots,m) \alpha_i^{(j)} = 0, j \in N_i. \quad (2)$$

Then, we define the multisplitting iterative method for the linear complementarity problem as follows:

Algorithm 2.1 (The multisplitting iterative method for the linear complementarity problem)

Step 1: Let $z^{(0)}$ be an arbitrary vector, $\varepsilon > 0$, $s(i,k), i = 1,2,\cdots,m$ and set $k = 0$.

Step 2: For each $l \in \{1,2,\cdots,s(i,k)\}$, let $y_i^{(l-1)} = z^{(k-1)}$.

Step 3: Parallel Calculate $\{y_i^{(l)}, i = 1,2,\cdots,m; l = 1,2,\cdots,s(i,k)\}$ where $y_i^{(l)}$ is the arbitrary solution:

$$\begin{align*}
y_i^{(l)} & \geq 0, \\
q - N_i z^{(k)} + M_i y_i^{(l)} & \geq 0, \\
(y_i^{(l)})^T (q - N_i z^{(k)} + M_i y_i^{(l)}) & = 0.
\end{align*} \quad (3)$$

Step 4: Calculate $E_i^{(k)}, i = 1,2,\cdots,m$, where

$$\begin{align*}
\min_{k_i} (z^T A z - 2q^T z) \\
z = \sum_{i=1}^{m} E_i^{(k)} y_i^{(s(i,k))}
\end{align*} \quad (4)$$
Step 5: Then, set
\[ z^{(k+1)} = \sum_{i=1}^{m} E_i^{(k)} y_i^{(i,k)} \] (5)

If \( \| Ax^{(k)} + q \|_2 < \varepsilon \) hold, then stop. Otherwise, let \( k := k + 1 \) and return to Step 2.

Convergence Theorems

In this section, we establish the convergence theory for Method 3.1 when the system matrix \( A \) of the \( LCP(q, A) \) is an \( H \)-matrix with positive diagonal entries.

**Lemma 3.1** [1] If \( A \in \mathbb{R}^{n \times n} \) is an \( H \)-matrix with positive diagonal elements positive definite, then the linear complementarity problem (1) has a unique solution for all vectors \( q \in \mathbb{R}^n \).

**Lemma 3.2** [1] If \( A \in \mathbb{R}^{n \times n} \) is positive definite, then the linear complementarity problem (1) has a unique solution for all vectors \( q \in \mathbb{R}^n \).

**Lemma 3.3** A matrix \( A \in \mathbb{R}^{n \times n} \) is a P-matrix if and only if the linear complementarity problem (1) has a unique solution for all vectors \( q \in \mathbb{R}^n \).

**Lemma 3.4** Let \( A \in \mathbb{R}^{n \times n} \) be a positive definite matrix, and let \( A = M - N \) be a P-regular splitting of \( A \). Then there exists \( r > 0 \) such that
\[ \| A^{1/2} M^{-1} N A^{-1/2} \|_2 \leq r < 1. \]

**Lemma 3.5** Let \( A = M_i - N_i (i = 1, 2, \cdots, m) \) a multisplitting of \( A \), where \( A \) and \( M_i \) are H-matrix with positive diagonal elements respective. Let \( z^* \) be a unique solution of the linear complementarity problem (1). Then, for an arbitrary vector \( q \in \mathbb{R}^n \) and any starting vector \( z^0 \in \mathbb{R}^n \), the uniquely defined sequence \( \{ z^k \} \), generated by Algorithm 2.1, satisfies
\[ < M_i z^k - z^* | N_i z^k - z^* > \]
(6)

**Proof.** To establish the expression (6), we first remark that \( y_i^{(i)} \) is uniquely defined because \( M_i (i = 1, 2, \cdots, m) \) is an H-matrix with positive diagonal elements (see Lemma 3.1). We verify (6) component by component. Consider an arbitrary index \( j \) and assume that
\[ | y_i^{(i)} - z^* | = (y_i^{(i)} - z^*) \]

Under this assumption, the inequality (6) holds clearly if \( (y_i^{(i)})_j = 0 \), because the \( j \)-th component of the left-hand vector in (6) is then nonpositive and the right-hand component is always nonnegative. Now, suppose that \( (y_i^{(i)})_j > 0 \). Then, according to \( y_i^{(i)} y^{(i, k+1)} \) be an arbitrary solution of the following linear complementarity problem (3), we have
\[ (q - N_i z^{(k)} + M_i y_i^{(i)})_j = 0. \]

On the other hand, we also have
\[ (q - N_i z^* + M_i y_i^{(i)})_j \geq 0. \]

Subtracting the last two expressions and rearranging terms, we deduce
\[ (q - N_i z^{(k)} + M_i y_i^{(i)})_j - (q - N_i z^* + M_i y_i^{(i)})_j \leq 0. \]

Therefore
\[(M_i (y^{(i)}_i - z^*))_j \leq (N_i (z^{(k)} - z^*))_j\]

which implies
\[< M_i > \mid y^{(i)}_i - z^* \mid \leq \mid N_i \mid \mid z^k - z^* \mid \]

(Because \( M_i \) is an H-matrix with positive diagonal elements and \( | y^{(i)}_i - z^* | = (y^{(i)}_i - z^*)_j \))

In a similar fashion, we may establish the same inequality (6) if \( | y^{(i)}_i - z^* | = (z^* - y^{(i)}_i)_j \). Consequently, the inequality (6) must hold.

**Theorem 3.6** Let \( A = M_i - N_i \) \((i = 1, 2, \ldots, m)\) a multisplitting of \( A \), where \( A \) and \( M_i \) are H-matrix with positive diagonal elements respective. Let \( z^* \) be a unique solution of the linear complementarity problem (1). Let \( E_i = \text{diag}(\alpha_i^{(1)}, \alpha_i^{(2)}, \ldots, \alpha_i^{(n)}) \in \mathbb{R}^{n \times n} (i = 1, 2, \ldots, m) \). Where \( \alpha_i^{(j)} = 0, j \in N_i \). If there exists \( A = M_{i_0} - N_{i_0} \) be a P-regular splitting of \( A \) and \( N_{i_0} = \phi \). Then, for any initial vector \( z^0 \), the iterative sequence \( \{z^k\} \), generated by the Algorithm 2.1 converges to the unique solution \( z^* \) of the linear complementarity problem (1).

**Proof.** Let \( z^* \) be the unique solution of the \( LCP(q, A) \), where the uniqueness comes from the assumption that \( A \) is an \( H \)-matrix with its diagonal elements. By the step 3 of the Algorithm 2.1 and Lemma 3.5, we have
\[ | y^{(i)}_i - z^* | \leq \mid M_i >^{-1}[ N_i \mid \mid z^k - z^* \mid . \]

By the expression (4) of the Algorithm 2.1, we have
\[ \min(z - z^*)A(z - z^*). \]

It follows from Lemma 3.5 that
\[ \| A^{1/2} (z^{(k)} - z^*) \|_2 \| A^{1/2} ((M_{i_0})^{-1} N_{i_0}) (z^{(k-1)} - z^*) \|_2 \leq \| A^{1/2} (< M_{i_0} >^{-1} N_{i_0}) A^{-1/2} \|_2 \| A^{1/2} (z^{(k-1)} - z^*) \|_2 \]
\[ \ldots \]
\[ \leq \| A^{1/2} (< M_{i_0} >^{-1} N_{i_0}) A^{-1/2} \|_2 \| A^{1/2} (z^{(0)} - z^*) \|_2 \]

We know from Lemma 3.4 that
\[ \| A^{1/2} (< M_{i_0} >^{-1} N_{i_0}) A^{-1/2} \|_2 \leq r < 1. \]

Therefore
\[ \lim_{k \to \infty} \| A^{1/2} (z^{(k)} - z^*) \|_2 = 0. \]

Thus
\[ \lim_{k \to \infty} \| z^{(k)} - z^* \|_2 = 0 \]

This completes the proof.

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References


