A Class of One-dimensional Chaotic Quadratic Curves

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Abstract. The paper studies the chaotic characteristics of a class of quadratic functions. In order to facilitate research, the Bézier control points are used to stretch the Logistic curve toward left-right or up-down. This family of quadratic curves are Li-Yorke chaos when certain conditions are met, it has been proved with Marotto snap-back-repeller’s theorem. In addition to this main results, the parameter points which are able to generate chaos are shown on the plane. From an application point of view, we will get more chaotic curves by moving the control points at both ends, that the chaotic curves can be used to generate chaotic sequences.

Introduction

According to parametric formula, the polynomial expressions are obtained as following:

\[
\begin{align*}
\begin{cases}
x(t) &= (1 - t)^2 x_0 + 2t(1 - t)x_1 + t^2 x_2 \\
y(t) &= (1 - t)^2 y_0 + 2t(1 - t)y_1 + t^2 y_2
\end{cases}
\end{align*}
\]

(0 ≤ t ≤ 1) (1)

(x0, y0), (x1, y1) and (x2, y2) are called Control-points.

When the three Control-points are (0, 0), (0.5, 2) and (1, 0) in Formula(1), the map is Logistic map

\[ y = 4x(1 - x). \]

In fact, lots of curves obtained by moving around the Logistic curve are chaotic.

We let the three points are (0, 0), (k1, k2) and (1, 0), for such Control-points, Formula(1) is changed to formula(2).

\[
\begin{align*}
\begin{cases}
x(t) &= 2t(1 - t)k_1 + t^2 \\
y(t) &= 2t(1 - t)k_2
\end{cases}
\end{align*}
\]

(2)

According to the following derivation, the formula(2) transformed to the Cartesian representation in Formula(3) and (4)

\[
t = \frac{-k_1 + \sqrt{4k_1^2 + 4(1 - 2k_1)x}}{1 - 2k_1}
\]

(3)

\[
y = f(x) = 2k_2 \times \frac{-k_1 + \sqrt{k_1^2 + (1 - 2k_1)x}}{1 - 2k_1} \times \frac{1 - k_1 + \sqrt{k_1^2 + (1 - 2k_1)x}}{(1 - 2k_1)}
\]

(4)

The main result of the paper is that system in Formula(4) is chaotic in the sense of Li-Yorke when some conditions are met.

While 0 ≤ x ≤ 1, \( k_1^2 + (1 - 2k_1)x > 0 \) (see formula(2)), the map in formula(4) is differentiable in \( R^1 \). Its derivative is
\[ y' = \frac{k_2}{(1 - 2k_1)} \left( \frac{1}{\sqrt{k_1^2 + (1 - 2k_1)x}} - 2 \right) \]

(5)

Mainly Results

The following Theorem 1 is the main result of the paper.

Theorem 1 \( f(x) \) in Formula(4) is chaotic in the sense of Li-Yorke, while

\[
0 < k_1 < 1 \quad k_1 \neq 0.5, \quad 1.9 < k_2 < 2 \quad \text{and} \quad \frac{12}{19} < N < \frac{19}{20}, \quad 1 - \frac{2}{k_2}N > 0,
\]

\[
1 - \frac{2}{k_2}P > 0, \quad \text{in which} \quad N = \frac{4k_2(k_2 - k_1)}{(2k_2 - 2k_1 + 1)^2}, \quad M = \frac{1}{2} - \frac{1 - 2k_1}{2k_2}N - \frac{1}{2} \sqrt{1 - \frac{2}{k_2}N},
\]

\[
P = \frac{1}{2} - \frac{1 - 2k_1}{2k_2}M + \frac{1}{2} \sqrt{1 - \frac{2}{k_2}M}.
\]

The following is the explanation and proof.

Step 1. About expanding fixed point

While

\[
x = f(x) = 2k_2 \times \left( -k_1 + \sqrt{k_1^2 + (1 - 2k_1)x} \right) \times \left[ 1 - \frac{k_1 + \sqrt{k_1^2 + (1 - 2k_1)x}}{(1 - 2k_1)} \right],
\]

The solution \( x = N = \frac{4k_2(k_2 - k_1)}{(2k_2 - 2k_1 + 1)^2} \) is the fixed point of \( f(x) \).

Let

\[
N = 2k_2 \times \left( -k_1 + \sqrt{k_1^2 + (1 - 2k_1)x} \right) \times \left[ 1 - \frac{k_1 + \sqrt{k_1^2 + (1 - 2k_1)x}}{(1 - 2k_1)} \right],
\]

\( M \) satisfied \( f(M) = M \) can be obtained,

Step 2. The sequence \( p^{(1)}, p^{(2)}, \ldots, p^{(n)}, \ldots \) have a limit

\[
p^{(n)} \quad \text{reads} \quad p^{(n)} = \frac{1}{2} - \frac{1 - 2k_1}{2k_2}p^{(n - 1)} + \frac{1}{2} \sqrt{1 - \frac{2}{k_2}p^{(n - 1)}}, \quad \text{then}
\]

\[
p^{(n)} - p^{(n-1)} = \frac{1 - 2k_1}{2k_2}(p^{(n-2)} - p^{(n-1)}) + \frac{1}{2} \left( \sqrt{1 - \frac{2}{k_2}p^{(n-1)}} - \sqrt{1 - \frac{2}{k_2}p^{(n-2)}} \right)
\]

\[
= \frac{1 - 2k_1}{2k_2}(p^{(n-2)} - p^{(n-1)}) + \frac{1}{k_2} \sqrt{1 - \frac{2}{k_2}p^{(n-1)}} - \sqrt{1 - \frac{2}{k_2}p^{(n-2)}},
\]

As long as the following inequality in (6) is satisfied, \( p^{(n)} \) is converges.

\[
\frac{1}{2k_2} + \frac{1}{k_2} \frac{1}{\sqrt{1 - \frac{2}{k_2}p^{(n-1)}} - \sqrt{1 - \frac{2}{k_2}p^{(n-2)}}} < u < 1
\]

(6)
Step 3. The limit of $P^{(n)}$ is $N$

See the front part of $P^{(n)}$, we have

$$
\lim_{n \to \infty} \left( \frac{1}{2} - \frac{1}{2} \times \frac{1 - 2k_1}{2k_2} + \frac{1}{2} \times \left( \frac{1 - 2k_1}{2k_2} \right)^2 - \frac{1}{2} \times \left( \frac{1 - 2k_1}{2k_2} \right)^3 + \cdots + (-1)^n \frac{1}{2} \times \left( \frac{1 - 2k_1}{2k_2} \right)^n \right) = \frac{1}{2} \left( 1 - \left( \frac{1 - 2k_1}{2k_2} \right)^n \right)
$$

Moreover, we find that

$$\lim_{n \to \infty} \sqrt{1 - \frac{2}{k_2} \left( \frac{1}{2} \times \frac{1 - 2k_1}{2k_2} \cdots + \frac{1}{2} \sqrt{1 - \frac{2}{k_2} \left( \cdots \right)} \right)} = \sqrt{1 - \frac{2}{k_2} P^{(n-1)}}$$

So that converges of formula (7) equivalents to its of sequence $P^{(x)}$.

Let $\lim_{n \to \infty} \sqrt{1 - \frac{2}{k_2} \left( \frac{1}{2} \times \frac{1 - 2k_1}{2k_2} \cdots + \frac{1}{2} \sqrt{1 - \frac{2}{k_2} \left( \cdots \right)} \right)} = Q$

$$\lim_{n \to \infty} P^{(x)} = \frac{k_2}{2k_2 - 2k_1 + 1} + \frac{k_2}{2k_2 - 2k_1 + 1} Q$$

So, the limit of $P^{(n)}$ is $N$.

Step 4. All eigenvalues of $DF(X)$ exceed 1 in norm for all $X \in B_+(Z)$.

For the one-dimensional map, eigenvalues of $DF(X)$ are just derivative of $F(X)$. In this paper, the norm is the absolute.

Easy to prove the absolute value of N’s derivative exceed 1.

$$y' = \left( \frac{1}{k_2} \left( \frac{1}{\sqrt{k_1^2 + (1 - 2k_1)x}} - 2 \right) \right) > 1$$

Step 5. $N$ is not a critical point,

$$\left| DF^{(k)}(N) \right| \neq 0$$

Using mathematical induction method, we can prove that multiplier factor isn’t equal to zero. Therefore, $N$ is not a critical point.

From step (1) to step (5), we conclude that while conditions of Theorem 1 is satisfied, there are snap-back-repellers for Formula(4), and Formula(4) is chaotic in the sense of Li-Yorke.

Summary

While $0 < k_1 < 1$ and $k_1 \neq 0.5$, $k_2 = 1.95$, a family of maps denoted by Formula(4) is chaotic.
The result of the paper can be used in the image encryption. A good encryption scheme should be sensitive to the secret keys, and the key space should be large enough to make brute-force attacks infeasible[7,8,9].

Further Study on the subject of the paper is:

Give out theoretical demonstration of that Formula(4) is Devaney chaos or not;
When left and right control points in Fig. 2 are changed, the map is or not chaotic;
If cubic Bézier curves are chaotic generated by four control points;
Whether or not Bézier surface are chaotic generated from nine or sixteen control points, etc.

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References