The Mallat Algorithm of Biorthogonal Two-direction Bivariate Wavelets and Application in Communication Engineering

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**Abstract.** In this paper we investigate the Mallat algorithms of biorthogonal bivariate two-direction wavelets. We introduce the concepts of biorthogonal bivariate two-direction refinement function and biorthogonal two-direction bivariate wavelet. The Mallat algorithms of biorthogonal two-direction bivariate wavelets was investigated. Moreover, a necessary and sufficient condition of complete reconstruction with decomposition is deduced. This algorithm is important for the decomposition and reconstruction of discrete signal with finite energy.

**Introduction**

Wavelet analysis has intently attracted many researchers’ attention in the last twenty years. It have been widely used in signal processing, image processing, voice recognition and synthesis, music, radar, machine fault diagnosis and monitoring, and so on [1–6]. Multiwavelets, which can offer properties as symmetry, orthogonality, short support at the same time, have been an active research field of wavelets analysis. To obtain some beautiful features, Yang and Li [7] introduce the concepts of two-direction scaling function and two-direction wavelets, give the definition of orthogonal two-direction scaling function and construct the orthogonal two-direction wavelets. Moreover, they investigate the approximation order of two-direction scaling functions. After the notion of two-direction wavelets is introduced, lots of researchers pay attention to the research of two-direction wavelets, such as the works [8, 9]. Two-direction scaling functions $\phi$ and two-direction wavelets $\psi$ associated with $\phi$, which are more general setting than the one-direction function and wavelets. The two-direction setting is more flexible than the one-direction setting. In the multiwavelet theory, the Mallat algorithms for two-direction bivariate wavelets is important. So, the notion of two-direction bi-orthogonal wavelets is introduced and to derive the decomposition and reconstruction algorithms biorhogonal two-direction wavelets is given in this paper.

**Notations and Fundamentals on Two-direction Wavelets**

We start from the following notations. $\mathbb{Z}$ and $\mathbb{R}$ denote all integers and all real numbers. We use $L^2(\mathbb{R}^2)$ denote the space of all square integrable functions on $\mathbb{R}^2$. Let $E$ is a $2 \times 2$ matrix whose all entries are integers and all eigenvalues is larger than one in modulus. The absolute value of the determinant of matrix $E$ is denote $m$, i.e., $|\det E|=m$. For two functions $f(x), g(x) \in L^2(\mathbb{R}^2)$, by $<f(x), g(x) >= \int_{\mathbb{R}^2} f(x) \overline{g(x)} dx$, where the $\overline{g(x)}$ means the complex conjugate.

In the following, two-direction scaling function is introduced. $\phi(x)$ is called two-direction scaling function, if $\phi(x) \in L^2(\mathbb{R}^2)$ satisfies the following two-direction scaling equation

$$
\phi(x) = \sum_{k \in \mathbb{Z}^2} p_k^+ \phi(Ex - k) + \sum_{k \in \mathbb{Z}^2} p_k^- \phi(k - Ex)
$$

(1)
Definition 1. We say that the two-direction scaling bivariate functions $\varphi(x)$ and $\tilde{\varphi}(x) \in L^2(R^2)$ is biorthogonal, if they satisfy:

$$<\varphi(x),\tilde{\varphi}(x-k)>=\delta_{0,k}, <\varphi(x),\tilde{\varphi}(k-x)>=0$$

(2)

where $\delta_{0,k}$ is the Kronecker symbol. Define a sequence of subspace $\{V_j\}_{j \in Z}$

\[ V_j = \text{Clos}_{L^2(R^2)}\{m^j\varphi(E^jx-k),m^j\varphi(1-E^jx):k,l \in Z^2,\}, \quad j \in Z \]

(3)

Definition 2. We say that trivariable function $\varphi(x)$ in Eq.(1) generates a two-direction multiresolution analysis $\{V_j\}_{j \in Z}$, if the sequence $\{V_j\}_{j \in Z}$ defined in (3) satisfies:
1) $V_j \subset V_{j+1}$; 2) $\bigcap_{j \in Z} V_j = [0,1]$, $V_j = L^2(R^2)$; 3) $\varphi(x) \in V_j \Leftrightarrow \varphi(Ex) \in V_{j+1}$; 4) The family $\{\varphi(x-k),\varphi(n-x):k,n \in Z^2\}$ is a Riesz basis of subspace $V_0$.

Corresponding to a two-direction scaling bivariate function $\varphi(x)$, a family trivariable functions $\psi_{\lambda}(x), \lambda \in \Lambda = \{1,2,\cdots,m-1\}$ is called a two-direction wavelets if $\{\psi_{\lambda}(x-k),\psi_{\lambda}(n-x):k,n \in Z^2, \lambda \in \Lambda\}$ forms a Riesz basis of subspace $W_0$, so that $V_i = V_0 \oplus W_0$.

\[ V_{j+1} = V_j \oplus W_j, \{m^j\psi_{\lambda}(E^jx-k),m^j\psi_{\lambda}(1-E^jx):k,l \in Z^2\} \]

forms a Riesz basis of $L^2(R^2)$.

Hence, a two-direction wavelets $\psi_{\lambda}(x)$ associated with a two-direction refinable function $\varphi(x)$ must satisfies the following equation

\[ \psi_{\lambda}(x) = \sum_{k \in Z^2} q_{\lambda,k}^+ \varphi(Ex-k) + \sum_{k \in Z^2} q_{\lambda,k}^- \varphi(k-Ex) \]

(4)

Definition 3. Let $\varphi(x), \tilde{\varphi}(x) \in L^2(R^2)$ be an biorthogonal two-direction scaling functions, and the two-direction wavelets $\psi_{\lambda}(x), \tilde{\psi}_{\lambda}(x), \lambda \in \Lambda$ satisfy the Eq. (4), where $\{q_{\lambda,k}^+\}_{k \in Z^2}$ and $\{q_{\lambda,k}^-\}_{k \in Z^2}$ are finitely supported sequences. If $\varphi(x), \tilde{\varphi}(x) \in L^2(R^2)$ and $\psi_{\lambda}(x), \tilde{\psi}_{\lambda}(x), \lambda \in \Lambda$ satisfy the following conditions

\[ \begin{aligned}
&<\varphi(x),\tilde{\psi}_{\lambda}(x-k)>=<\varphi(x),\tilde{\psi}_{\lambda}(k-x)>=0 \\
&<\tilde{\varphi}(x),\psi_{\lambda}(x-k)>=<\tilde{\varphi}(x),\psi_{\lambda}(k-x)>=0 \\
&<\psi_{\lambda}(x),\psi_{\mu}(x-k)>=\delta_{\lambda,\mu}\delta_{0,k} \\
&<\tilde{\psi}_{\lambda}(x),\tilde{\psi}_{\mu}(x-k)>=0
\end{aligned} \]

(5)

Like Eq. (1) and Eq. (4), there are following equations

\[ \tilde{\varphi}(x) = \sum_{k \in Z^2} \tilde{p}_k^+ \tilde{\varphi}(Ex-k) + \sum_{k \in Z^2} \tilde{p}_k^- \tilde{\varphi}(k-Ex) \]

(6)

\[ \psi_{\lambda}(x) = \sum_{k \in Z^2} \tilde{p}_{\lambda,k}^+ \tilde{\varphi}(Ex-k) + \sum_{k \in Z^2} \tilde{p}_{\lambda,k}^- \tilde{\varphi}(k-Ex), \lambda \in \Lambda \]

(7)

Where $\{\tilde{p}_k^+\}_{k \in Z^2}, \{\tilde{p}_k^-\}_{k \in Z^2}$, $\{\tilde{p}_{\lambda,k}^+\}_{k \in Z^2}$ and $\{\tilde{p}_{\lambda,k}^-\}_{k \in Z^2}$ are finitely supported sequences.

Denote

\[ \begin{aligned}
\psi_{\lambda,j,n}(x) &= m^{j/2}\psi_{\lambda}(E^jx-n), \quad \psi_{\lambda,j,n}(x) = m^{j/2}\psi_{\lambda}(n-E^jx), \\
\phi_{\lambda,n}(x) &= m^{j/2}\varphi(E^jx-n), \quad \phi_{\lambda,n}(x) = m^{j/2}\varphi(n-E^jx)
\end{aligned} \]

\[ \begin{aligned}
\tilde{\psi}_{\lambda,j,n}(x) &= m^{j/2}\tilde{\psi}_{\lambda}(E^jx-n), \quad \tilde{\psi}_{\lambda,j,n}(x) = m^{j/2}\tilde{\psi}_{\lambda}(n-E^jx), \\
\tilde{\phi}_{\lambda,n}(x) &= m^{j/2}\tilde{\varphi}(E^jx-n), \quad \tilde{\phi}_{\lambda,n}(x) = m^{j/2}\tilde{\varphi}(n-E^jx)
\end{aligned} \]
\[ \tilde{\phi}^+_{j,n}(x) = m^{j/2}\tilde{\phi}(E^j x - n), \quad \tilde{\phi}^-_{j,n}(x) = m^{j/2}\tilde{\phi}(n - E^j x) \]

The Main Results

In this part, we will proceed to study the decomposition and reconstruction Mallat algorithm of biorthogonal two-direction trivariable wavelets and present a necessary and sufficient condition of complete reconstruction.

For energy finite signal \( f(x) \in L^2(R^2) \), the approximation \( f_{j+1} \) of signal \( f \) at the scale \( m^{j+1} \) will respectively be piecewise constant and piecewise affine on \( V_j \). Because of \( V_{j+1} = V_j \oplus W_j = V_j \oplus W_j^{(1)} \oplus W_j^{(2)} \oplus \cdots \oplus W_j^{(m-1)}, j \in Z \), here the spaces \( W_j^{(\lambda)} = \text{clos}_{E(R^2)}[\psi_{-j,n}^{(\lambda)}, \psi_{-j,n}^{(\lambda)} : n \in Z^2], \lambda \in \Lambda, j \in Z \). So \( f_{j+1} \) can represent as following

\[ f_{j+1}(x) = \sum_{j,k} c^+_{j+1,k} \varphi^+_{j+1,k}(x) + \sum_{j,k} c^-_{j+1,k} \varphi^-_{j+1,k}(x) \]

Furthermore, the right side of Eq.(8) can be expressed as following

\[ \sum_{j,k} c^+_{j+1,k} \varphi^+_{j+1,k}(x) + \sum_{j,k} c^-_{j+1,k} \varphi^-_{j+1,k}(x) = \sum_{j,k} d^+_{j+1,k} \psi^+_{j+1,k}(x) + \sum_{j,k} d^-_{j+1,k} \psi^-_{j+1,k}(x) \]

In the following, we give the decomposition algorithms of biorthogonal two-direction wavelets.

**Theorem 1.** Let \( f_{j+1} \in V_{j+1} \), then we have following formula

\[ c^+_{j,j} = m^{-1/2} \sum_{k \in Z^2} (c^+_{j+1,k} \tilde{p}^+_k + c^-_{j+1,k} \tilde{p}^-_k) \]

\[ c^-_{j,j} = m^{-1/2} \sum_{k \in Z^2} (c^+_{j+1,k} \tilde{p}^+_k + c^-_{j+1,k} \tilde{p}^-_k) \]

\[ d^+_{j,j} = m^{-1/2} \sum_{k \in Z^2} (d^+_{j+1,k} \tilde{q}^+_k + d^-_{j+1,k} \tilde{q}^-_k) \]

\[ d^-_{j,j} = m^{-1/2} \sum_{k \in Z^2} (d^+_{j+1,k} \tilde{q}^+_k + d^-_{j+1,k} \tilde{q}^-_k) \]

**Proof.** To make an inner product on both ends of Eq. (8) by \( \tilde{\phi}^+_{j,j} \), we have

\[ c^+_{j,j} = \sum_k c^+_{j+1,k} < \varphi^+_{j+1,k}(x), \tilde{\phi}^+_{j,j} > + \sum_k c^-_{j+1,k} < \varphi^-_{j+1,k}(x), \tilde{\phi}^-_{j,j} > \]

Inspired by Eq.(1), we have

\[ < \varphi^+_{j+1,k}(x), \tilde{\phi}^+_{j,j} > = m^{-1/2} \tilde{p}^+_k, \quad < \varphi^-_{j+1,k}(x), \tilde{\phi}^-_{j,j} > = m^{-1/2} \tilde{p}^-_k \]

Substitute the Eq.(15) into Eq.(14) , we obtain the Eq. (9). Similarly, the Eq. (10) can be obtained. The following we prove the Eq. (12) and Eq. (13).

To make an inner product on both ends of Eq. (8) and Eq. (9)by \( \tilde{\psi}^+_{j,j}(x) \), we have

\[ d^+_{j,j} = \sum_k c^+_{j+1,k} < \varphi^+_{j+1,k}(x), \tilde{\psi}^+_{j,j} > + \sum_k c^-_{j+1,k} < \varphi^-_{j+1,k}(x), \tilde{\psi}^-_{j,j} > \]

Inspired by Eq.(1), we have

\[ < \varphi^+_{j+1,k}(x), \tilde{\psi}^+_{j,j} > = m^{-1/2} \tilde{q}^+_k, \quad < \varphi^-_{j+1,k}(x), \tilde{\psi}^-_{j,j} > = m^{-1/2} \tilde{q}^-_k \]

Thus, we obtain the Eq. (12). Similarly, the Eq. (13) can be obtained.

In the following, we give the reconstruction algorithms of biorthogonal two-direction bivariate wavelets.

**Theorem 2.** Let \( f_{j+1} \in V_{j+1} \), then we have following formula
\[ c^+_{j+1} = m^{-1/2} \sum_{k} (c^+_{j,k} p^-_{k-\ell,k} + c^-_{j,k} p^+_{\ell,k}) + m^{-1/2} \sum_{\lambda \in \Lambda} \sum_{k} (d^+_{\lambda,j,k} q^-_{\lambda,k-\ell,k} + d^-_{\lambda,j,k} q^+_{\lambda,k}) \]  
\[ c^-_{j+1} = m^{-1/2} \sum_{k} (c^+_{j,k} p^-_{k-\ell,k} + c^-_{j,k} p^+_{\ell,k}) + m^{-1/2} \sum_{\lambda \in \Lambda} \sum_{k} (d^+_{\lambda,j,k} q^-_{\lambda,k-\ell,k} + d^-_{\lambda,j,k} q^+_{\lambda,k}) \]  

**Proof.** Firstly, we prove the Eq. (18). To make an inner product on both ends of Eq. (8) and Eq.(9) by \( \tilde{\phi}^+_{j+1} \), we have

\[ c^+_{j+1} = \sum_{k} c^+_{j,k} <\phi^+_{j+1,k}(x), \tilde{\phi}^+_{j+1}(x)> + \sum_{k} c^-_{j,k} <\phi^+_{j+1,k}(x), \tilde{\phi}^-_{j+1}(x)> + \sum_{\lambda \in \Lambda} \sum_{k} d^+_{\lambda,j,k} <\psi^+_{\lambda,j,k}(x), \tilde{\phi}^+_{j+1}(x)> + \sum_{\lambda \in \Lambda} \sum_{k} d^-_{\lambda,j,k} <\psi^+_{\lambda,j,k}(x), \tilde{\phi}^-_{j+1}(x)> \]

Inspired by Eq.(1), we have

\[ <\phi^+_{j,k}(x), \phi^+_{j+1,k}(x)> = m^{-1/2} p^+_{\ell,k} \cdot <\phi^+_{j,k}(x), \tilde{\phi}^+_{j+1}(x)> = m^{-1/2} p^+_{\ell,k} \]

Thus, we obtain the Eq. (18). Similarly, the Eq. (19) can be obtained.

In following, we discuss the conditions of complete reconstruction after signal decomposition according to the Mallat algorithm.

**Theorem 3.** Let \( \phi(x), \tilde{\phi}(x) \) be an biorthogonal two-direction scaling functions, and the two-direction wavelets \( \psi_{j,k}(x), \tilde{\psi}_{j,k}(x) \) associated with \( \phi(x), \tilde{\phi}(x) \), then the sufficient and necessary condition of complete reconstruction as follows

\[
\sum_{k \in \mathbb{Z}^2} (\tilde{p}_{\ell,k}^+ p^-_{\ell,k} + \tilde{p}_{\ell,k}^- p^+_{\ell,k}) = m \delta_{0,0} \cdot \sum_{k \in \mathbb{Z}^2} (\tilde{q}_{\ell,k}^+ q^-_{\ell,k} + \tilde{q}_{\ell,k}^- q^+_{\ell,k}) = m \delta_{0,0} \delta_{\lambda,\mu} \\
\sum_{k \in \mathbb{Z}^2} (\tilde{p}_{\ell,k}^- p^+_{\ell,k} + \tilde{p}_{\ell,-k}^- p^+_{\ell,-k}) = 0, \sum_{k \in \mathbb{Z}^2} (\tilde{q}_{\ell,k}^- q^+_{\ell,k} + \tilde{q}_{\ell,-k}^- q^+_{\ell,-k}) = 0 \\
\sum_{k \in \mathbb{Z}^2} (\tilde{q}_{\ell,k}^+ q^-_{\ell,k} + \tilde{q}_{\ell,-k}^+ q^-_{\ell,-k}) = 0, \sum_{k \in \mathbb{Z}^2} (\tilde{p}_{\ell,k}^+ q^-_{\ell,k} + \tilde{p}_{\ell,-k}^+ q^-_{\ell,-k}) = 0 \\
\sum_{k \in \mathbb{Z}^2} (\tilde{p}_{\ell,k}^+ q^-_{\ell,k} + \tilde{p}_{\ell,k}^- q^+_{\ell,k}) = 0, \sum_{k \in \mathbb{Z}^2} (\tilde{q}_{\ell,k}^+ q^-_{\ell,k} + \tilde{q}_{\ell,k}^- q^+_{\ell,k}) = 0 \\
\sum_{k \in \mathbb{Z}^2} (\tilde{p}_{\ell,k}^- q^+_{\ell,k} + \tilde{p}_{\ell,-k}^- q^+_{\ell,-k}) = 0, \sum_{k \in \mathbb{Z}^2} (\tilde{q}_{\ell,k}^- q^+_{\ell,k} + \tilde{q}_{\ell,-k}^- q^+_{\ell,-k}) = 0 \\
\sum_{k \in \mathbb{Z}^2} (\tilde{p}_{\ell,k}^- q^+_{\ell,k} + \tilde{p}_{\ell,k}^- q^+_{\ell,k}) = 0, \sum_{k \in \mathbb{Z}^2} (\tilde{q}_{\ell,k}^- q^+_{\ell,k} + \tilde{q}_{\ell,k}^- q^+_{\ell,k}) = 0
\]

**Proof.** From the definition of biorthogonality, for all \( n \in \mathbb{Z}^2 \), we have

\[ \delta_{0,n} = <\tilde{\phi}(x-n), \phi(x)> = \sum_{k \in \mathbb{Z}^2} \tilde{p}_{\ell,k}^- \phi(E(x-n) - k) = \sum_{k \in \mathbb{Z}^2} \tilde{p}_{\ell,k}^- \phi[k - E(x-n)] + \sum_{k \in \mathbb{Z}^2} \tilde{p}_{\ell,k}^- \phi(E(x-n) - k) \]

so the identity \( \sum_{k \in \mathbb{Z}^2} (\tilde{p}_{\ell,k}^- p^+_{\ell,k} + \tilde{p}_{\ell,k}^- p^+_{\ell,k}) = m \delta_{0,0} \) is holds. Similarly, the other can be obtained.

Thus, Theorem 3 is proved.

Mallat algorithm does not need to know the concrete structure expression of scale function and wavelet function. Only need to know the filter coefficients corresponding to the scale function and wavelet function. By using this set of filter coefficients, the wavelet decomposition and reconstruction of the signal can be realized, and because of this, Mallat algorithm in the field of signal processing has been widely used.
Summary
The notion of two-direction biorthogonal bivariate two-direction wavelets is introduced. Mattlat algorithm of two-direction biorthogonal bivariate two-direction wavelets has been proposed. A necessary and sufficient condition of complete reconstruction with decomposition are established.

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References